An. Şt. Univ. Ovidius Constanţa Vol. 19(1), 2011, 331–346

Some unified algorithms for finding minimum norm fixed point of nonexpansive semigroups in Hilbert spaces

Yonghong Yao, Yeong-Cheng Liou

Abstract

In this paper, we introduce two general algorithms (one implicit and one explicit) for finding a common fixed point of a nonexpansive semigroup $\{T(s)\}_{s\geq 0}$ in Hilbert spaces. We prove that both approaches converge strongly to a common fixed point of $\{T(s)\}_{s\geq 0}$. Such common fixed point x^* is the unique solution of some variational inequality, which is the optimality condition for some minimization problem. As special cases of the above two algorithms, we obtain two schemes which both converge strongly to the minimum norm common fixed point of $\{T(s)\}_{s\geq 0}$.

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall a mapping $f : C \to H$ is called to a contraction if, for all $x, y \in C$, there exists $\rho \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \rho\|x - y\|$. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx-Ty\| \leq \|x-y\|, \forall x, y \in C$. Denote the set of fixed points of T by $Fix(T)$. Let A be a strongly positive bounded linear operator on H, i.e., there exists a constant $\bar{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2$ for all $x \in H$.

Key Words: Common fixed point; Variational inequality; Nonexpansive Semigroup; Algorithms; Minimum norm.
Mathematics Subject Classification: 47H05; 47H10; 47H17.
Received: January, 2010
Accepted: December, 2010

Iterative methods for nonexpansive mappings are widely used to solve convex minimization problems, see, for instance, [1],[2], [4]-[6], [9],[11]-[14],[16]-[30], [32], [33]. A typical problem is to minimize a function over the set of fixed points of a nonexpansive mapping T,

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle. \tag{1.1}$$

In [27], Xu proved that the sequence $\{x_n\}$ defined by $x_{n+1} = \alpha_n b + (1 - \alpha_n A)Tx_n, n \geq 0$ strongly converges to the unique solution of (1.1) under certain conditions. Recently, Marino and Xu [17] introduced the viscosity approximation method $x_{n+1} = \alpha_n \gamma f(x_n) + (1-\alpha_n A)Tx_n, n \geq 0$ and proved that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \forall x \in Fix(T)$$

which is the optimality condition for the minimization problem

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - h(x)$$

where h is a potential function for γf (i.e., $h^{'} = \gamma f$ on H).

In this paper, we focus on nonexpansive semigroup $\{T(s)\}_{s\geq 0}$. Recall that a family $S := \{T(s)\}_{s\geq 0}$ of mappings of C into itself is called a nonexpansive semigroup if it satisfies the following conditions:

(S1) $T(0)x = x$ for all $x \in C$;

(S2) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(S3) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(S4) for all $x \in H$, $s \to T(s)x$ is continuous.

We denote by $Fix(T(s))$ the set of fixed points of $T(s)$ and by $Fix(S)$ the set of all common fixed points of S, i.e. $Fix(S) = \bigcap_{s\geq 0} Fix(T(s))$. It is known that $Fix(S)$ is closed and convex (Lemma 1 in [1]).

Algorithms for nonexpansive semigroups have been considered by some authors, please consult [3], [7],[8],[10],[15], [31]. The following interesting problem arises: Can one construct some more general algorithms which unify the above algorithms?

On the other hand, we also notice that it is quite often to seek a particular solution of a given nonlinear problem, in particular, the minimum-norm solution. For instance, given a closed convex subset C of a Hilbert space H_1 and a bounded linear operator $R : H_1 \to H_2$, where H_2 is another Hilbert space. The C-constrained pseudoinverse of R, $R_C^\dagger$, is then defined as the minimum-norm solution of the constrained minimization problem

$$R_C^\dagger(b) := \arg\min_{x \in C} \|Rx - b\|$$

which is equivalent to the fixed point problem

$$x = P_C(x - \lambda R^*(Rx - b))$$

where P_C is the metric projection from H_1 onto C, R^* is the adjoint of R, $\lambda > 0$ is a constant, and $b \in H_2$ is such that $P_{\overline{R(C)}}(b) \in R(C)$.

It is therefore another interesting problem to invent some algorithms that can generate schemes which converge strongly to the minimum-norm solution of a given problem.

In this paper, we introduce two general algorithms (one implicit and one explicit) for finding a common fixed point of a nonexpansive semigroup $\{T(s)\}_{s \geq 0}$ in Hilbert spaces. We prove that both approaches converge strongly to a common fixed point of $\{T(s)\}_{s \geq 0}$. Such common fixed point x^* is the unique solution of some variational inequality, which is the optimality condition for some minimization problem. As special cases of the above two algorithms, We obtain two schemes which both converge strongly to the minimum norm common fixed point of $\{T(s)\}_{s \geq 0}$.

2 Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \forall x, y \in H.$$

Moreover, P_C is characterized by the following properties:

$$\langle x - P_C x, y - P_C x \rangle \leq 0, \tag{2.1}$$

and

$$\|x-y\|^2 \geq \|x-P_Cx\|^2+\|y-P_Cx\|^2,$$

for all $x \in H$ and $y \in C$.

We need the following lemmas for proving our main results.

Lemma 2.1. *([23]) Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\{T(s)\}_{s\geq 0}$ be a nonexpansive semigroup on C. Then, for every $h \geq 0$,*

$$\lim_{t\to\infty} sup_{x\in C}\Big\|\frac{1}{t}\int_0^t T(s)xds - T(h)\frac{1}{t}\int_0^t T(s)xds\Big\| = 0.$$

Lemma 2.2. *([12]) Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I-S)x_n \to y$ strongly, then $(I-S)x^* = y$.*

Lemma 2.3. ([18]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\gamma_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1-\gamma_n)x_n + \gamma_n y_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_n - y_{n-1}\| - \|x_n - x_{n-1}\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.4. *([26]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(2) $\limsup_{n\to\infty}\delta_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section we will show our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S = \{T(s)\}_{s\geq 0} : C \to C$ be a nonexpansive semigroup with $Fix(S) \neq \emptyset$. Let $f : C \to H$ be a ρ-contraction (possibly non-self). Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\bar{\gamma} > 0$. Let $\{\lambda_t\}_{0<t<1}$ be a continuous net of positive real numbers such that $\lim_{t\to 0} \lambda_t = +\infty$. Let γ and β be two real numbers such that $0 < \gamma < \bar{\gamma}/\rho$ and $\beta \in [0, 1)$. Let the net $\{x_t\}$ be defined by the following implicit scheme:*

$$x_t = P_C[t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds], t \in (0, 1). \quad (3.1)$$

Then, as $t \to 0+$, the net $\{x_t\}$ strongly converges to $x^* \in Fix(S)$ which is the unique solution of the following variational inequality:

$$x^* \in Fix(S), \quad \langle(\gamma f - A)x^*, x - x^*\rangle \leq 0, \quad \forall x \in Fix(S). \quad (3.2)$$

In particular, if we take $f = 0$ and $A = I$, then the net $\{x_t\}$ defined by (3.1) reduces to

$$x_t = P_C[\beta x_t + (1-\beta-t)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds], t \in (0, 1). \quad (3.3)$$

In this case, the net $\{x_t\}$ defined by (3.3) converges in norm to the minimum norm fixed point x^* of $Fix(S)$, namely, the point x^* is the unique solution to the minimization problem:

$$x^* = \arg\min_{x\in Fix(S)} \|x\|. \quad (3.4)$$

Proof. First, we note that the net $\{x_t\}$ defined by (3.1) is well-defined. We define a mapping

$$Gx := P_C[t\gamma f(x) + \beta x + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x ds], t \in (0, 1).$$

It follows that

$$\begin{aligned}
\|Gx - Gy\| &\leq \|t\gamma(f(x) - f(y)) + \beta(x-y) + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t}(T(s)x - T(s)y)ds\| \\
&\leq t\gamma\|f(x) - f(y)\| + \beta\|x-y\| + \|((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t}(T(s)x - T(s)y)ds\| \\
&\leq t\gamma\rho\|x-y\| + \beta\|x-y\| + (1-\beta-t\bar{\gamma})\|x-y\| \\
&= (1-(\bar{\gamma}-\gamma\rho)t)\|x-y\|.
\end{aligned}$$

This implies that the mapping G is a contraction and so it has a unique fixed point. Therefore, the net $\{x_t\}$ defined by (3.1) is well-defined.

Take $p \in Fix(S)$. By (3.1), we have

$$\begin{aligned}
\|x_t - p\| &= \|P_C[t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds] - p\| \\
&\leq \|t(\gamma f(x_t) - Ap) + \beta(x_t - p) + ((1-\beta)I - tA)(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p)\| \\
&\leq t\|\gamma f(x_t) - Ap\| + \beta\|x_t - p\| + (1-\beta-\bar{\gamma}t)\frac{1}{\lambda_t}\int_0^{\lambda_t} \|T(s)x_t - T(s)p\| ds \\
&\leq t\gamma\|f(x_t) - f(p)\| + t\|\gamma f(p) - Ap\| + \beta\|x_t - p\| + (1-\beta-\bar{\gamma}t)\|x_t - p\| \\
&\leq t\gamma\rho\|x_t - p\| + t\|\gamma f(p) - Ap\| + \beta\|x_t - p\| + (1-\beta-\bar{\gamma}t)\|x_t - p\|.
\end{aligned}$$

It follows that

$$\|x_t - p\| \leq \frac{1}{\bar{\gamma} - \gamma\rho}\|\gamma f(p) - Ap\|$$

which implies that the net $\{x_t\}$ is bounded.

Set $R := \frac{1}{\bar{\gamma}-\gamma\rho}\|\gamma f(p) - Ap\|$. It is clear that $\{x_t\} \subset B(p, R)$. Notice that

$$\|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p\| \leq \|x_t - p\| \leq R.$$

Moreover, we observe that if $x \in B(p, R)$ then

$$\|T(s)x - p\| \leq \|T(s)x - T(s)p\| \leq \|x - p\| \leq R,$$

i.e., $B(p, R)$ is $T(s)$-invariant for all s.

Set $y_t = t\gamma f(x_t) + \beta x_t + ((1-\beta)I - tA)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds$. From (3.1), we deduce

$$\begin{aligned}
\|T(\tau)x_t - x_t\| &= P_C[T(\tau)x_t] - P_C[y_t]\| \\
&\leq \|T(\tau)x_t - y_t\| \\
&\leq \|T(\tau)x_t - T(\tau)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| \\
&\quad +\|T(\tau)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| \\
&\quad +\|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - y_t\| \\
&\leq \|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| + \|T(\tau)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - \\
&\quad -\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| + \|\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - y_t\| \\
&\leq \frac{2t}{1-\beta}\|\gamma f(x_t) - \frac{A}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\| \\
&\quad +\|T(\tau)\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\|.
\end{aligned}$$

By Lemma 2.1, we deduce for all $0 \leq \tau < \infty$

$$\lim_{t\to 0}\|T(\tau)x_t - x_t\| = 0. \tag{3.5}$$

Note that $x_t = P_C[y_t]$. By using the property of the metric projection (2.1), we have

$$\begin{aligned}
\|x_t - p\|^2 &= \langle x_t - y_t, x_t - p\rangle + \langle y_t - p, x_t - p\rangle \\
&\leq \langle y_t - p, x_t - p\rangle \\
&= t\langle \gamma f(x_t) - Ap, x_t - p\rangle + \beta\|x_t - p\|^2 \\
&\quad +\langle((1-\beta)I - tA)(\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds - p), x_t - p\rangle \\
&\leq \beta\|x_t - p\|^2 + (1-\beta-\bar{\gamma}t)\|x_t - p\|^2 \\
&\quad +t\gamma\langle f(x_t) - f(p), x_t - p\rangle + t\langle \gamma f(p) - Ap, x_t - p\rangle \\
&\leq [1-(\bar{\gamma}-\gamma\rho)t]\|x_t - p\|^2 + t\langle \gamma f(p) - Ap, x_t - p\rangle.
\end{aligned}$$

Therefore,

$$\|x_t - p\|^2 \leq \frac{1}{\bar{\gamma}-\gamma\rho}\langle \gamma f(p) - Ap, x_t - p\rangle, \forall p \in Fix(S).$$

From this inequality, we have immediately that $\omega_w(x_t) = \omega_s(x_t)$, where $\omega_w(x_t)$ and $\omega_s(x_t)$ denote the set of weak and strong cluster points of $\{x_t\}$, respectively.

Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$, $y_n := y_{t_n}$ and $\lambda_n := \lambda_{t_n}$. Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $x^* \in C$. Also $y_n \to x^*$ weakly. Noticing (3.5) we can use Lemma 2.2 to get $x^* \in Fix(S)$.

We can rewrite (3.1) as

$$(A - \gamma f)x_t = -\frac{1}{t}((1-\beta)I - tA)[x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds] + \frac{1}{t}(x_t - y_t).$$

Therefore,

$$\begin{aligned}\langle (A-\gamma f)x_t, x_t - p\rangle &= -\frac{1-\beta}{t}[\frac{1}{\lambda_t}\int_0^{\lambda_t}\langle (I-T(s))x_t - (I-T(s))p, x_t - p\rangle ds] \\ &\quad + \frac{1}{\lambda_t}\langle A\int_0^{\lambda_t}[x_t - T(s)x_t]ds, x_t - p\rangle + \frac{1}{t}\langle x_t - y_t, x_t - p\rangle.\end{aligned}$$

Noting that $I - T(s)$ is monotone and $\langle x_t - y_t, x_t - p\rangle \le 0$, so

$$\begin{aligned}\langle (A-\gamma f)x_t, x_t - p\rangle &\le \frac{1}{\lambda_t}\langle A\int_0^{\lambda_t}[x_t - T(s)x_t]ds, x_t - p\rangle \\ &= \langle Ax_t - \frac{A}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds, x_t - p\rangle \\ &\le \|A\|\|x_t - \frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\|\|x_t - p\| \\ &\le \frac{t}{1-\beta}\|A\|\|\gamma f(x_t) - A\frac{1}{\lambda_t}\int_0^{\lambda_t} T(s)x_t ds\|\|x_t - p\|.\end{aligned}$$

Taking the limit through $t := t_{n_i} \to 0$, we have

$$\langle (A-\gamma f)x^*, x^* - p\rangle = \lim_{i\to\infty}\langle (A-\gamma f)x_{n_i}, x_{n_i} - p\rangle \le 0.$$

Since the solution of the variational inequality (3.2) is unique. Hence $\omega_w(x_t) = \omega_s(x_t)$ is singleton. Therefore, $x_t \to x^*$.

In particular, if we take $f = 0$ and $A = I$, then it follows that $x_t \to x^* = P_{Fix(S)}(0)$, which implies that x^* is the minimum norm fixed point of S. As a matter of fact, by (3.2), we deduce

$$\langle x^*, x^* - x\rangle \le 0, \quad \forall x \in Fix(S),$$

that is,

$$\|x^*\|^2 \leq \langle x^*, x\rangle \leq \|x^*\|\|x\|, \quad \forall x \in Fix(S).$$

Therefore, the point x^* is the unique solution to the minimization problem

$$x^* = \arg\min_{x\in Fix(S)} \|x\|.$$

This completes the proof. □

Next we introduce an explicit algorithm for finding a solution of minimization problem (3.4). This scheme is obtained by discretizing the implicit scheme (3.1). We will show the strong convergence of this algorithm.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S = \{T(s)\}_{s\geq 0} : C \to C$ be a nonexpansive semigroup with $Fix(S) \neq \emptyset$. Let $f : C \to H$ be a ρ-contraction (possibly non-self) with $\rho \in [0, 1)$. Let A be a strongly positive linear bounded self-adjoint operator on H with coefficient $\bar{\gamma} > 0$. Let γ and β be two real numbers such that $0 < \gamma < \bar{\gamma}/\rho$ and $\beta \in [0, 1)$. Let the sequence $\{x_n\}$ be generated iteratively by the following explicit algorithm:*

$$x_{n+1} = (1-\gamma_n)x_n+\gamma_n P_C[\alpha_n\gamma f(x_n)+\beta x_n+((1-\beta)I-\alpha_n A)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds], n \geq 0, \tag{3.6}$$

where $\{\alpha_n\}$ and $\{\gamma_n\}$ are real number sequence in $[0, 1]$ and $\{\lambda_n\}$ is a positive real number. Suppose that the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $\lim_{n\to\infty} \lambda_n = \infty$ *and* $\lim_{n\to\infty} \frac{\lambda_n-\lambda_{n-1}}{\lambda_n} = 0$*;*

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Then the sequence $\{x_n\}$ strongly converges to $x^* \in Fix(S)$ which is the unique solution of the variational inequality (3.2).

In particular, if we take $f = 0$ and $A = I$, then the sequence $\{x_n\}$ generated by (3.6) *reduces to*

$$x_{n+1} = (1-\gamma_n)x_n+\gamma_n P_C[\beta x_n+(1-\alpha_n-\beta)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds], n \geq 0. \tag{3.7}$$

In this case, the sequence $\{x_n\}$ converges in norm to the minimum norm fixed point x^* of $Fix(S)$.

Proof. Take $p \in Fix(S)$. From (3.6), we have

$$\begin{aligned} \|x_{n+1} - p\| &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\Big(\alpha_n\|\gamma f(x_n) - Ap\| + \beta\|x_n - p\| \\ &\quad +(1-\beta-\bar{\gamma}\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n}\|T(s)x_n - T(s)p\|ds\Big) \\ &\le (1-\gamma_n)\|x_n - p\| + \gamma_n\Big(\alpha_n\gamma\rho\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + \beta\|x_n - p\| \\ &\quad +(1-\beta-\alpha_n\bar{\gamma})\|x_n - p\|\Big) \\ &= [1-(\bar{\gamma}-\rho\gamma)\alpha_n\gamma_n]\|x_n - p\| + \alpha_n\gamma_n\|\gamma f(p) - Ap\|. \end{aligned}$$

It follows that by induction

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma}-\gamma\rho}\}.$$

Set $y_n = P_C[\alpha_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)z_n]$ for all $n \ge 0$, where $z_n = \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds$. Hence, we have

$$\begin{aligned} \|y_n - y_{n-1}\| &\le \|\alpha_n\gamma f(x_n) + \beta x_n + ((1-\beta)I - \alpha_n A)z_n \\ &\quad -\alpha_{n-1}\gamma f(x_{n-1}) - \beta x_{n-1} - ((1-\beta)I - \alpha_{n-1}A)z_{n-1}\| \\ &= \|\gamma\alpha_n(f(x_n) - f(x_{n-1})) + \gamma(\alpha_n - \alpha_{n-1})f(x_{n-1}) + \beta(x_n - x_{n-1}) \\ &\quad +((1-\beta)I - \alpha_n A)(z_n - z_{n-1}) + (\alpha_{n-1} - \alpha_n)Az_{n-1}\| \\ &\le \gamma\alpha_n\|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|(\|\gamma f(x_{n-1})\| + \|Az_{n-1}\|) \\ &\quad +\beta\|x_n - x_{n-1}\| + (1-\beta-\alpha_n\bar{\gamma})\|z_n - z_{n-1}\| \end{aligned}$$

and

$$\begin{aligned} \|z_n - z_{n-1}\| &= \|\frac{1}{\lambda_n}\int_0^{\lambda_n}[T(s)x_n - T(s)x_{n-1}]ds + (\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}})\int_0^{\lambda_{n-1}} T(s)x_{n-1}ds \\ &\quad +\frac{1}{\lambda_n}\int_{\lambda_{n-1}}^{\lambda_n} T(s)x_{n-1}ds\| \\ &\le \frac{1}{\lambda_n}\int_0^{\lambda_n}\|T(s)x_n - T(s)x_{n-1}\|ds + \frac{1}{\lambda_n}\|\int_{\lambda_{n-1}}^{\lambda_n}[T(s)x_{n-1} - T(s)p]ds\| \\ &\quad +|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}|\int_0^{\lambda_{n-1}}\|T(s)x_{n-1} - T(s)p\|ds \\ &\le \|x_n - x_{n-1}\| + \frac{2|\lambda_n - \lambda_{n-1}|}{\lambda_n}\|x_{n-1} - p\|. \end{aligned}$$

Therefore,

$$\begin{aligned}\|y_n - y_{n-1}\| &\leq \gamma\alpha_n\rho\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|\gamma f(x_{n-1})\| + \|Az_{n-1}\|) \\ &\quad +\beta\|x_n - x_{n-1}\| + (1 - \beta - \alpha_n\bar{\gamma})\|x_n - x_{n-1}\| \\ &\quad +\frac{2|\lambda_n - \lambda_{n-1}|}{\lambda_n}\|x_{n-1} - p\| \\ &\leq [1 - (\bar{\gamma} - \gamma\rho)\alpha_n]\|x_n - x_{n-1}\| + M(|\alpha_n - \alpha_{n-1}| + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n}),\end{aligned}$$

where $M > 0$ is a constant such that

$$\sup_n\{\|\gamma f(x_{n-1})\| + \|Az_{n-1}\|, 2\|x_{n-1} - p\|\} \leq M.$$

Hence, we get

$$\limsup_{n\to\infty}(\|y_n - y_{n-1}\| - \|x_n - x_{n-1}\|) \leq 0.$$

This together with Lemma 2.3 imply that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Therefore,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\gamma_n\|y_n - x_n\| = 0.$$

Note that

$$\begin{aligned}\|T(\tau)x_n - x_n\| &\leq \|T(\tau)x_n - T(\tau)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\| \\ &\quad +\|T(\tau)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\| \\ &\quad +\|\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds - x_n\| \\ &\leq \|T(\tau)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\| \\ &\quad +2\|x_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\|. \qquad (3.8)\end{aligned}$$

From (3.6), we have

$$\begin{aligned}\|x_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\| \\ &\leq \|x_n - x_{n+1}\| + (1-\gamma_n)\|x_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\| \\ &\quad +\gamma_n\alpha_n\gamma\|f(x_n)\| + \gamma_n\beta\|x_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\| \\ &\quad +\gamma_n\alpha_n\|A\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - \frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\| &\leq \frac{1}{(1-\beta)\gamma_n}\Big\{\|x_n - x_{n+1}\| + \gamma_n\alpha_n\gamma\|f(x_n)\| \\ &\quad +\gamma_n\alpha_n\|A\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds\|\Big\} \\ &\to 0. \end{aligned} \tag{3.9}$$

From (3.8), (3.9) and Lemma 2.1, we have

$$\lim_{n\to\infty} \|T(\tau)x_n - x_n\| = 0 \text{ for every } \tau \geq 0. \tag{3.10}$$

Notice that $\{x_n\}$ is a bounded sequence. Let $\tilde{x}$ be a weak limit of $\{x_n\}$. Putting $x^* = P_{Fix(S)}(I - A + \gamma f)$. Then there exists R such that $B(x^*, R)$ contains $\{x_n\}$. Moreover, $B(x^*, R)$ is $T(s)$-invariant for every $s \geq 0$; therefore, without loss of generality, we can assume that $\{T(s)\}_{s\geq 0}$ is a nonexpansive semigroup on $B(x^*, R)$. By the demiclosedness principle (Lemma 2.2) and (3.10), we have $\tilde{x} \in Fix(S)$. Therefore,

$$\limsup_{n\to\infty}\langle\gamma f(x^*) - Ax^*, x_{n+1} - x^*\rangle = \lim_{n\to\infty}\langle\gamma f(x^*) - Ax^*, \tilde{x} - x^*\rangle \leq 0.$$

Finally, we prove $x_n \to x^*$. Set $u_n = t\gamma f(x_n)+\beta x_n+((1-\beta)I-tA)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)x_n ds$. It follows that $y_n = P_C[u_n]$ for all $n \geq 0$. By using the property of the metric projection (2.1), we have

$$\langle y_n - u_n, y_n - x^*\rangle \leq 0.$$

So,

$$\begin{aligned}
\|y_n - x^*\|^2 &= \langle y_n - x^*, y_n - x^*\rangle \\
&= \langle y_n - u_n, y_n - x^*\rangle + \langle u_n - x^*, y_n - x^*\rangle \\
&\le \langle u_n - x^*, y_n - x^*\rangle \\
&= \alpha_n\gamma\langle f(x_n) - f(x^*), y_n - x^*\rangle + \alpha_n\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle \\
&\quad +\beta\langle x_n - x^*, y_n - x^*\rangle + \langle((1-\beta)I - \alpha_n A)(z_n - x^*), y_n - x^*\rangle \\
&\le \alpha_n\gamma\|f(x_n) - f(x^*)\|\|y_n - x^*\| + \alpha_n\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle \\
&\quad +\beta\|x_n - x^*\|\|y_n - x^*\| + (1-\beta-\bar{\gamma}\alpha_n)\|z_n - x^*\|\|y_n - x^*\| \\
&\le [1-(\bar{\gamma}-\gamma\rho)\alpha_n]\|x_n - x^*\|\|y_n - x^*\| + \alpha_n\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle \\
&\le \frac{1-(\bar{\gamma}-\gamma\rho)\alpha_n}{2}\|x_n - x^*\|^2 + \frac{1}{2}\|y_n - x^*\| + \alpha_n\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle,
\end{aligned}$$

that is,

$$\|y_n - x^*\|^2 \le [1-(\bar{\gamma}-\gamma\rho)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle.$$

By the convexity of the norm, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|y_n - x^*\|^2 \\
&\le [1-(\bar{\gamma}-\gamma\rho)\alpha_n\gamma_n]\|x_n - x^*\|^2 + 2\alpha_n\gamma_n\langle \gamma f(x^*) - Ax^*, y_n - x^*\rangle.
\end{aligned}$$

Hence, all conditions of Lemma 2.4 are satisfied. Therefore, we immediately deduce that $x_n \to x^*$.
In particular, if we take $f = 0$ and $A = I$, then it is clear that $x^* = P_{Fix(S)}(0)$ is the unique solution to the minimization problem $x^* = \arg\min_{x\in Fix(S)} \|x\|$.
This completes the proof. □

Acknowledgment

The first author was supported in part by Colleges and Universities Science and Technology Development Foundation (20091003) of Tianjin and NSFC 11071279. The second author was supported in part by NSC 99-2221-E-230-006.

References

[1] F. E. Browder, Convergence of approximation to fixed points of nonexpansive nonlinear mappings in Hilbert spaces, *Arch. Rational Mech. Anal.* 24 (1967), 82-90.

[2] F. E. Browder, Convergence theorems for sequences of nonlinear operators in Banach spaces, *Math. Z.* 100 (1967), 201-225.

[3] N. Buong, Strong convergence theorem for nonexpansive semigroups in Hilbert space, *Nonlinear Anal.* 72 (2010), 4534-4540.

[4] L.C. Ceng, P. Cubiotti and J.C. Yao, Strong convergence theorems for finitely many nonexpansive mappings and applications, *Nonlinear Anal.* 67 (2007), 1464-1473.

[5] S.S. Chang, Viscosity approximation methods for a finite family of nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* 323 (2006), 1402-1416.

[6] S.S. Chang, J.C. Yao, J.K. Kim and L. Yang, Iterative approximation to convex feasibility problems in Banach space, *Fixed Point Theory and Applications* 2007 (2007), Article ID 46797, 19 pagesdoi:10.1155/2007/46797.

[7] R. Chen and H. He, Viscosity approximation of common fixed points of nonexpansive semigroups in Banach space, *Appl. Math. Lett.* 20 (2007), 751-757.

[8] R. Chen and Y. Song, Convergence to common fixed point of nonexpansive semigroups, *J. Comput. Appl. Math.* 200 (2007), 566-575.

[9] Y.J. Cho and X. Qin, Convergence of a general iterative method for nonexpansive mappings in Hilbert spaces, *J. Comput. Appl. Math.* 228 (2009), no. 1, 458-465.

[10] F. Cianciaruso, G. Marino and L. Muglia, Iterative methods for equilibrium and fixed point problems for nonexpansive semigroups in Hilbert spaces, *J. Optim. Theory Appl.* 146 (2010), 491-509.

[11] Y.L. Cui and X. Liu, Notes on Browder's and Halpern's methods for nonexpansive maps, *Fixed Point Theory* 10 (2009), no. 1, 89-98.

[12] K. Geobel and W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, 1990.

[13] K. Goebel and S. Reich, Uniform convexity, hyperbolic geometry, and nonexpansive mappings, Marcel Dekker, 1984.

[14] B. Halpern, Fixed points of nonexpansive maps, *Bull. Am. Math. Soc.* 73 (1967), 957-961.

[15] A.T. Lau and W. Takahashi, Fixed point properties for semigroup of nonexpansive mappings on Fréchet spaces. *Nonlinear Anal.* 70 (2009), no. 11, 3837-3841.

[16] P. L. Lions, Approximation de points fixes de contractions, *C.R. Acad. Sci. Sèr. A-B Paris* 284(1977), 1357-1359.

[17] G. Marino, H.K Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* 318 (2006), 43-52.

[18] A. Moudafi, Viscosity approximation methods for fixed-points problems, *J. Math. Anal. Appl.* 241 (2000), 46-55.

[19] N. Nadezhkina and W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.* 128 (2006), 191-201.

[20] Z. Opial, Weak convergence of the sequence of successive approximations of nonexpansive mappings, *Bull. Amer. Math. Soc.* 73 (1967), 595-597.

[21] A. Petruşel and J. C. Yao, Viscosity approximation to common fixed points of families of nonexpansive mappings with generalized contractions mappings, *Nonlinear Anal.* 69 (2008), 1100-1111.

[22] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.* 75 (1980), 287-292

[23] T. Shimizu, W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, *J. Math. Anal. Appl.* 211 (1997), 71-83.

[24] T. Suzuki, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Amer. Math. Soc.*, 135 (2007), 99-106.

[25] H.K. Xu, Another control condition in an iterative method for nonexpansive mappings, *Bull. Austral. Math. Soc.* 65 (2002), 109-113.

[26] H.K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.* 66 (2002), 240-256.

[27] H.K. Xu, An iterative approach to quadratic optimization, *J. Optim. Theory Appl.* 116 (2003), 659-678.

[28] Y. Yao, Y.C. Liou and G. Marino, Strong convergence of two iterative algorithms for nonexpansive mappings in Hilbert spaces, *Fixed Point Theory Appl.*, 2009 (2009), Article ID 279058, 7 pages, doi:10.1155/2009/279058.

[29] Y. Yao, Y.C. Liou and J.C. Yao, An iterative algorithm for approximating convex minimization problem, *Appl. Math. Comput.* 188 (2007), 648-656.

[30] Y. Yao and J.C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, *Appl. Mathe. Comput.*, 186 (2007), 1551-1558.

[31] H. Zegeye and N. Shahzad, Strong convergence theorems for a finite family of nonexpansive mappings and semigroups via the hybrid method, *Nonlinear Anal.* 72 (2010), 325-329.

[32] L.C. Zeng and J.C. Yao, Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings, *Nonlinear Anal.* 64 (2006), 2507-2515.

[33] H. Zhou, L. Wei and Y.J. Cho, Strong convergence theorems on an iterative method for a family of finite nonexpansive mappings in reflexive Banach spaces, *Appl. Math. Comput.* 173 (2006), 196-212.

Department of Mathematics, Tianjin Polytechnic University,
Tianjin 300160, People's Republic of China
e-mail: yaoyonghong@yahoo.cn

Department of Information Management,
Cheng Shiu University, Kaohsiung 833, Taiwan,
e-mail: simplex_liou@hotmail.com