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The Vertex Algebra $M(1)^+$ and Certain Affine Vertex Algebras of Level -1

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Abstract. We give a coset realization of the vertex operator algebra $M(1)^+$ with central charge ℓ. We realize $M(1)^+$ as a commutant of certain affine vertex algebras of level -1 in the vertex algebra $L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)\otimes L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)$. We show that the simple vertex algebra $L_{C_\ell^{(1)}}(-\Lambda_0)$ can be (conformally) embedded into $L_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$ and find the corresponding decomposition. We also study certain coset subalgebras inside $L_{C_\ell^{(1)}}(-\Lambda_0)$.

Key words: vertex operator algebra; affine Kac–Moody algebra; coset vertex algebra; conformal embedding; $\mathcal{W}$-algebra

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1 Introduction

In the last few years various types of $\mathcal{W}$-algebras have been studied in the framework of vertex operator algebras (see [3, 4, 9, 13, 26]). In this paper we will be focused on $\mathcal{W}$-algebras which admit coset realization. To any vertex algebra V and its subalgebra U, one can associate a new vertex algebra

$$\mathrm{Com}(U,V)=\{v\in V \mid u_n v=0 \text{ for all } u\in U,\ n\geq 0\}$$

called the *commutant* (or *coset*) of U in V. This is a very important construction in the theory of vertex operator algebras, because it gives a realization of a large family of $\mathcal{W}$-algebras. Another important construction is the *orbifold* construction, where a new vertex operator algebra is obtained as invariants in a given vertex operator algebra with respect to the finite automorphism group. As we shall see in our paper, some vertex algebras admit both realizations, coset and orbifold.

In this paper we consider certain coset vertex algebras for vertex algebras associated to affine Lie algebras. Let $\mathfrak{g}$ be a simple Lie algebra of type X_n, $\hat{\mathfrak{g}}$ the associated affine Lie algebra of type $X_n^{(1)}$, and $L_{X_n^{(1)}}(k\Lambda_0)$ the simple vertex operator algebra associated to $\hat{\mathfrak{g}}$ of level $k\in\mathbb{C}$, $k\neq -h^\vee$. For $k,m\in\mathbb{Z}_{>0}$, $L_{X_n^{(1)}}((k+m)\Lambda_0)$ is a subalgebra of $L_{X_n^{(1)}}(k\Lambda_0)\otimes L_{X_n^{(1)}}(m\Lambda_0)$, and one has the associated coset vertex operator algebra

$$\mathrm{Com}(L_{X_n^{(1)}}((k+m)\Lambda_0), L_{X_n^{(1)}}(k\Lambda_0)\otimes L_{X_n^{(1)}}(m\Lambda_0)). \tag{1}$$

Although there are no precise general results (to the best of our knowledge) about the structure of these cosets, it is believed that these vertex operator algebras are finitely generated and rational.

In [7], we present a vertex-algebraic proof of the fact that in the case $k = 1$ and affine Lie algebras of types $D_n^{(1)}$ and $B_n^{(1)}$, the coset (1) is isomorphic to the rational vertex operator algebra V_L^+.

The situation is even more complicated for general $k, m \in \mathbb{C}$, such that $k, m, k + m \neq -h^\vee$. Then one has the coset vertex operator algebra

$$\mathrm{Com}\big(\widetilde{L}_{X_n^{(1)}}((k+m)\Lambda_0), L_{X_n^{(1)}}(k\Lambda_0) \otimes L_{X_n^{(1)}}(m\Lambda_0)\big),$$

where $\widetilde{L}_{X_n^{(1)}}((k+m)\Lambda_0)$ is a certain affine vertex operator algebra associated to $X_n^{(1)}$ of level $k+m$ (not necessarily simple). In this paper we identify some special cases of such cosets, and it turns out that they are not rational.

The construction in [7] is based on fermionic construction of vertex algebras and certain conformal embeddings. In the present paper we use bosonic construction of vertex algebras and construct new conformal embeddings of affine vertex algebras at level -1. By applying the bosonic realization of the affine vertex algebras $L_{A_1^{(1)}}(-\frac{1}{2}\Lambda_0)$ and $L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)$ (cf. [17]) we consider coset vertex algebras

$$\begin{aligned}
&\mathrm{Com}\big(L_{A_1^{(1)}}(-\Lambda_0), L_{A_1^{(1)}}(-\tfrac{1}{2}\Lambda_0) \otimes L_{A_1^{(1)}}(-\tfrac{1}{2}\Lambda_0)\big) \qquad \text{and} \\
&\mathrm{Com}\big(\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0), L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0) \otimes L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0)\big).
\end{aligned} \tag{2}$$

It is interesting that these cosets have central charge 1. We show that these cosets are isomorphic to $M(1)^+$, where $M(1)$ is the Heisenberg vertex operator algebra of rank 1, and $M(1)^+$ is the $\mathbb{Z}_2$-orbifold vertex algebra studied in [15]. The structure theory of $M(1)^+$ shows that these cosets are irrational vertex operator algebras and isomorphic to $W(2,4)$-algebra with central charge $c = 1$.

By combining results from [1] and the present paper, we classify irreducible ordinary $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$-modules. We believe that the (tensor) category of $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$-modules is related to the (tensor) category of $M(1)^+$-modules. We plan to address this correspondence in our forthcoming publications.

Generalizing (2), we use a natural realization of the vertex operator algebra $L_{A_1^{(1)}}(-\Lambda_0)^{\otimes \ell}$ as a subalgebra of $L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0) \otimes L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)$, and prove that

$$\mathrm{Com}\big(L_{A_1^{(1)}}(-\Lambda_0)^{\otimes \ell}, L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0) \otimes L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0)\big)$$

is isomorphic to $M(1)^+$, where $M(1)$ is the Heisenberg vertex operator algebra of rank ℓ.

Our construction is based on a new, interesting conformal embedding of affine vertex operator algebras at level -1 which can be of independent interest. We show that $L_{C_\ell^{(1)}}(-\Lambda_0)$ is conformally embedded into $L_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$ and that

$$L_{A_{2\ell-1}^{(1)}}(-\Lambda_0) \cong L_{C_\ell^{(1)}}(-\Lambda_0) \oplus L_{C_\ell^{(1)}}(-2\Lambda_0 + \Lambda_2),$$

which implies that $L_{C_\ell^{(1)}}(-\Lambda_0)$ is a $\mathbb{Z}_2$-orbifold of $L_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$.

By using conformal embeddings we study certain categories of $A_{2\ell-1}^{(1)}$-modules of level -1 from [8] as $C_\ell^{(1)}$-modules. It turns out that irreducible highest weight $A_{2\ell-1}^{(1)}$-modules $L_{A_{2\ell-1}^{(1)}}(-(n+1)\Lambda_0 + n\Lambda_1)$ and $L_{A_{2\ell-1}^{(1)}}(-(n+1)\Lambda_0 + n\Lambda_{2\ell-1})$ ($n \in \mathbb{Z}_{>0}$) are also irreducible as $C_\ell^{(1)}$-modules. This result is an affine analogue of the isomorphism of finite-dimensional C_ℓ-modules:

$$V_{A_{2\ell-1}}(n\omega_{2\ell-1}) \cong V_{A_{2\ell-1}}(n\omega_1) \cong V_{C_\ell}(n\omega_1).$$

Using these conformal embeddings we also show that the coset

$$\mathrm{Com}\big(L_{A_1^{(1)}}(-\Lambda_0)^{\otimes \ell}, L_{C_\ell^{(1)}}(-\Lambda_0)\big)$$

is isomorphic to $M(1)^+$, where $M(1)$ is the Heisenberg vertex operator algebra of rank $\ell-1$.

2 Preliminaries

Let V be a vertex algebra [10, 19, 20, 27]. For a subalgebra U of V denote by

$$\mathrm{Com}(U,V) = \{v \in V \mid u_n v = 0 \text{ for all } u \in U,\ n \geq 0\}$$

the commutant of U in V (cf. [21, 22, 27]). Then, $\mathrm{Com}(U,V)$ is a subalgebra of V (also called coset vertex algebra).

Let $\mathfrak{h}$ be a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$, considered as an Abelian Lie algebra. Let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$ be its affinization with the center K. Then the free bosonic Fock space $M(1) = S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}])$ is a simple vertex operator algebra of central charge $\ell = \dim \mathfrak{h}$, with Virasoro vector

$$\omega = \frac{1}{2}\sum_{i=1}^{\ell} h^{(i)}(-1)^2\mathbf{1},$$

where $\{h^{(1)},\ldots,h^{(\ell)}\}$ is any orthonormal basis of $\mathfrak{h}$ (cf. [20, 27]). We shall also use the notation $M_{\mathfrak{h}}(1)$ to emphasize the associated vector space $\mathfrak{h}$.

Vertex algebra $M(1)$ has an order 2 automorphism which is lifted from the map $h \mapsto -h$, for $h \in \mathfrak{h}$. Denote by $M(1)^+$ (or $M_{\mathfrak{h}}(1)^+$) the subalgebra of invariants of that automorphism. The irreducible modules for $M(1)^+$ were classified in [15] and [16]. For $\ell = 1$, it was proved in [12] that $M(1)^+$ is generated by ω and one primary vector of conformal weight 4, so it is isomorphic to a $W(2,4)$-algebra with central charge 1.

Let $\mathfrak{g}$ be the simple Lie algebra of type X_n, and $\hat{\mathfrak{g}}$ the associated affine Lie algebra of type $X_n^{(1)}$. For any weight Λ of $\hat{\mathfrak{g}}$, denote by $L_{X_n^{(1)}}(\Lambda)$ the irreducible highest weight $\hat{\mathfrak{g}}$-module. Denote by Λ_i, $i = 0,\ldots,n$ the fundamental weights of $\hat{\mathfrak{g}}$ (cf. [23]). We shall also use the notation $V_{X_n}(\mu)$ for a highest weight $\mathfrak{g}$-module of highest weight μ, and ω_i, $i = 1,\ldots,n$ for the fundamental weights of $\mathfrak{g}$.

For any $k \in \mathbb{C}$, denote by $N_{X_n^{(1)}}(k\Lambda_0)$ the generalized Verma $\hat{\mathfrak{g}}$-module with highest weight $k\Lambda_0$. Then, $N_{X_n^{(1)}}(k\Lambda_0)$ is a vertex operator algebra of central charge $\frac{k \dim \mathfrak{g}}{k+h^\vee}$, for any $k \neq -h^\vee$, with Virasoro vector obtained by Sugawara construction:

$$\omega = \frac{1}{2(k+h^\vee)}\sum_{i=1}^{\dim \mathfrak{g}} a^i(-1)b^i(-1)\mathbf{1}, \tag{3}$$

where $\{a^i\}_{i=1,\ldots,\dim \mathfrak{g}}$ is an arbitrary basis of $\mathfrak{g}$, and $\{b^i\}_{i=1,\ldots,\dim \mathfrak{g}}$ the corresponding dual basis of $\mathfrak{g}$ with respect to the symmetric invariant bilinear form, normalized by the condition that the length of the highest root is $\sqrt{2}$ (cf. [18, 21, 24, 27, 28]).

It follows that any quotient of $N_{X_n^{(1)}}(k\Lambda_0)$ is a vertex operator algebra, for $k \neq -h^\vee$. Specially, $L_{X_n^{(1)}}(k\Lambda_0)$ is a simple vertex operator algebra, for any $k \neq -h^\vee$.

3 Simple Lie algebras of type C_ℓ and $A_{2\ell-1}$

Consider two 2ℓ-dimensional vector spaces $A_1 = \oplus_{i=1}^{2\ell} \mathbb{C} a_i^+$, $A_2 = \oplus_{i=1}^{2\ell} \mathbb{C} a_i^-$ and let $A = A_1 \oplus A_2$. The Weyl algebra $W_{2\ell}$ is the complex associative algebra generated by A with non-trivial relations

$$[a_i^+, a_j^-] = \delta_{i,j}, \qquad 1 \le i, j \le 2\ell.$$

The normal ordering on A is defined by

$$:xy: = \frac{1}{2}(xy + yx), \qquad x, y \in A.$$

Define

$$e^A_{\epsilon_i - \epsilon_j} = :a_i^+ a_j^-:, \qquad f^A_{\epsilon_i - \epsilon_j} = :a_j^+ a_i^-:, \qquad 1 \le i, j \le 2\ell, \quad i < j,$$

and

$$H_i = - :a_i^+ a_i^-:, \qquad 1 \le i \le 2\ell.$$

Then the Lie algebra $\mathfrak{g}_1$ generated by the set

$$\left\{ e^A_{\epsilon_i - \epsilon_j}, f^A_{\epsilon_i - \epsilon_j} \mid 1 \le i, j \le 2\ell, \ i < j \right\}$$

is the simple Lie algebra of type $A_{2\ell-1}$ (cf. [11] and [17]). The Cartan subalgebra $\mathfrak{h}_1$ is spanned by

$$\{ H_i - H_{i+1} \mid 1 \le i \le 2\ell - 1 \}.$$

Let θ be the automorphism of $W_{2\ell}$ of order two given by

$$a_i^+ \mapsto a^-_{2\ell+1-i}, \qquad a^-_{2\ell+1-i} \mapsto a_i^+, \qquad a_i^- \mapsto -a^+_{2\ell+1-i}, \qquad a^+_{2\ell+1-i} \mapsto -a_i^-,$$

for $i = 1, \dots, \ell$. Clearly, $\mathfrak{g}_1$ is θ-stable and

$$\theta(e^A_{\epsilon_i - \epsilon_j}) = -e^A_{\epsilon_{2\ell+1-j} - \epsilon_{2\ell+1-i}}, \qquad \theta(e^A_{\epsilon_i - \epsilon_{2\ell+1-j}}) = e^A_{\epsilon_j - \epsilon_{2\ell+1-i}},$$

and similarly for root vectors associated to negative roots.

The subalgebra $\mathfrak{g}$ of $\mathfrak{g}_1$ generated by

$$e_{2\epsilon_i} = a_i^+ a^-_{2\ell+1-i}, \qquad f_{2\epsilon_i} = a_i^- a^+_{2\ell+1-i},$$
$$e_{\epsilon_i + \epsilon_j} = \frac{1}{2}(a_i^+ a^-_{2\ell+1-j} + a_j^+ a^-_{2\ell+1-i}), \qquad f_{\epsilon_i + \epsilon_j} = \frac{1}{2}(a_i^- a^+_{2\ell+1-j} + a_j^- a^+_{2\ell+1-i}),$$
$$e_{\epsilon_i - \epsilon_j} = \frac{1}{2}(a_i^+ a_j^- - a^+_{2\ell+1-j} a^-_{2\ell+1-i}), \qquad f_{\epsilon_i - \epsilon_j} = \frac{1}{2}(a_j^+ a_i^- - a^+_{2\ell+1-i} a^-_{2\ell+1-j}),$$

for $i, j = 1, \dots, \ell$, $i < j$, is the simple Lie algebra of type C_ℓ. The Cartan subalgebra $\mathfrak{h}$ is spanned by

$$\{ H_i - H_{2\ell+1-i} \mid 1 \le i \le \ell \}.$$

Clearly, θ acts as 1 on $\mathfrak{g}$. Furthermore,

$$e^*_{\epsilon_1 + \epsilon_2} = \frac{1}{2}(a_1^+ a^-_{2\ell-1} - a_2^+ a^-_{2\ell}) \in \mathfrak{g}_1$$

is a highest weight vector for $\mathfrak{g}$, which generates the irreducible $\mathfrak{g}$-module $V_{C_\ell}(\omega_2)$. Clearly, θ acts as -1 on $V_{C_\ell}(\omega_2)$. We obtain the decomposition

$$\mathfrak{g}_1 \cong \mathfrak{g} \oplus V_{C_\ell}(\omega_2). \tag{4}$$

Since

$$\dim V_{A_{2\ell-1}}(n\omega_1) = \dim V_{C_\ell}(n\omega_1),$$

one easily concludes that the irreducible $\mathfrak{g}_1$-module $V_{A_{2\ell-1}}(n\omega_1)$ remains irreducible when restricted to $\mathfrak{g}$. Thus,

$$V_{A_{2\ell-1}}(n\omega_1) \cong V_{C_\ell}(n\omega_1) \qquad \text{for} \quad n \in \mathbb{Z}_{\geq 0}. \tag{5}$$

Similarly

$$V_{A_{2\ell-1}}(n\omega_{2\ell-1}) \cong V_{C_\ell}(n\omega_1) \qquad \text{for} \quad n \in \mathbb{Z}_{\geq 0}. \tag{6}$$

We will consider certain affine analogues of relations (4)–(6).

In what follows we shall need the following decompositions of $\mathfrak{g}$-modules:

$$\begin{aligned} V_{C_\ell}(\omega_2) \otimes V_{C_\ell}(\omega_2) &\cong V_{C_\ell}(2\omega_2) \oplus V_{C_\ell}(\omega_1+\omega_3) \oplus V_{C_\ell}(\omega_4) \\ &\quad \oplus V_{C_\ell}(2\omega_1) \oplus V_{C_\ell}(\omega_2) \oplus V_{C_\ell}(0) \quad (\ell \geq 4), \\ V_{C_3}(\omega_2) \otimes V_{C_3}(\omega_2) &\cong V_{C_3}(2\omega_2) \oplus V_{C_3}(\omega_1+\omega_3) \oplus V_{C_3}(2\omega_1) \oplus V_{C_3}(\omega_2) \oplus V_{C_3}(0), \\ V_{C_2}(\omega_2) \otimes V_{C_2}(\omega_2) &\cong V_{C_2}(2\omega_2) \oplus V_{C_2}(2\omega_1) \oplus V_{C_2}(0). \end{aligned} \tag{7}$$

4 Weyl vertex algebras and symplectic affine Lie algebras

The Weyl algebra $W_\ell(\frac{1}{2}+\mathbb{Z})$ is a complex associative algebra generated by

$$a_i^{\pm}(r), \qquad r \in \tfrac{1}{2}+\mathbb{Z}, \quad 1 \leq i \leq \ell$$

with non-trivial relations

$$[a_i^+(r), a_j^-(s)] = \delta_{r+s,0}\delta_{i,j},$$

where $r, s \in \frac{1}{2}+\mathbb{Z}$, $i, j \in \{1, \dots, \ell\}$.

Let M_ℓ be the irreducible $W_\ell(\frac{1}{2}+\mathbb{Z})$-module generated by the cyclic vector $\mathbf{1}$ such that

$$a_i^{\pm}(r)\mathbf{1} = 0 \qquad \text{for} \quad r > 0, \quad 1 \leq i \leq \ell.$$

Define the following fields on M_ℓ

$$a_i^{\pm}(z) = \sum_{n\in\mathbb{Z}} a_i^{\pm}(n+\tfrac{1}{2}) z^{-n-1}.$$

The fields $a_i^{\pm}(z)$, $i = 1, \dots, \ell$ generate on M_ℓ the unique structure of a simple vertex algebra (cf. [18, 24]). Let us denote the corresponding vertex operator by Y.

We have the following Virasoro vector in M_ℓ:

$$\omega = \frac{1}{2}\sum_{i=1}^{\ell} \left(a_i^-(-\tfrac{3}{2})a_i^+(-\tfrac{1}{2}) - a_i^+(-\tfrac{3}{2})a_i^-(-\tfrac{1}{2}) \right)\mathbf{1}. \tag{8}$$

Let $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$. Then M_ℓ is $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded with respect to $L(0)$:

$$M_\ell := \bigoplus_{m\in\frac{1}{2}\mathbb{Z}_{\geq 0}} M_\ell(m), \qquad M_\ell(m) = \{v \in M_\ell \mid L(0)v = mv\}.$$

Note that $M_\ell(0) = \mathbb{C}\mathbf{1}$. For $v \in M_\ell(m)$ we shall write $\mathrm{wt}(v) = m$.

The following result is well-known.

Theorem 1 ([17]). *We have*

$$M_\ell \cong L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0) \bigoplus L_{C_\ell^{(1)}}(-\tfrac{3}{2}\Lambda_0 + \Lambda_1).$$

In the case $\ell = 1$ we have

$$M_1 \cong L_{A_1^{(1)}}(-\tfrac{1}{2}\Lambda_0) \bigoplus L_{A_1^{(1)}}(-\tfrac{3}{2}\Lambda_0 + \Lambda_1).$$

Remark 1. The highest weights of modules from Theorem 1 are admissible in the sense of [25]. Representations of vertex operator algebras associated to affine Lie algebras of type $A_1^{(1)}$ and $C_\ell^{(1)}$ with admissible highest weights were studied in [5] and [2].

We shall now consider the vertex algebra $M_{2\ell}$ and its subalgebra $L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0) \otimes L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)$. For $\ell = 1$, $L_{A_1^{(1)}}(-\frac{1}{2}\Lambda_0) \otimes L_{A_1^{(1)}}(-\frac{1}{2}\Lambda_0)$ is a subalgebra of M_2.

Let $\theta : M_{2\ell} \to M_{2\ell}$ be the automorphism of order two of the vertex algebra $M_{2\ell}$ which is lifted from the following automorphism of the Weyl algebra

$$\begin{aligned} a_i^+(s) &\mapsto a_{2\ell+1-i}^-(s), & a_{2\ell+1-i}^-(s) &\mapsto a_i^+(s), \\ a_i^-(s) &\mapsto -a_{2\ell+1-i}^+(s), & a_{2\ell+1-i}^+(s) &\mapsto -a_i^-(s), \end{aligned}$$

for $i = 1, \dots, \ell$ and $s \in \frac{1}{2} + \mathbb{Z}$.

If we have a subalgebra $U \subset M_{2\ell}$ which is θ-stable, we define

$$U^0 = \{u \in U \mid \theta(u) = u\}, \qquad U^1 = \{u \in U \mid \theta(u) = -u\}.$$

Define the following vectors in $M_{2\ell}$:

$$\begin{aligned} b_i^+ &= \frac{1}{\sqrt{2}}(a_i^+(-\tfrac{1}{2}) + a_{2\ell+1-i}^-(-\tfrac{1}{2}))\mathbf{1}, & b_{2\ell+1-i}^+ &= \frac{\sqrt{-1}}{\sqrt{2}}(a_i^+(-\tfrac{1}{2}) - a_{2\ell+1-i}^-(-\tfrac{1}{2}))\mathbf{1}, \\ b_i^- &= \frac{1}{\sqrt{2}}(a_i^-(-\tfrac{1}{2}) - a_{2\ell+1-i}^+(-\tfrac{1}{2}))\mathbf{1}, & b_{2\ell+1-i}^- &= \frac{-\sqrt{-1}}{\sqrt{2}}(a_i^-(-\tfrac{1}{2}) + a_{2\ell+1-i}^+(-\tfrac{1}{2}))\mathbf{1}, \end{aligned}$$

for $i = 1, \dots, \ell$. Then the subalgebra generated by b_i^+, b_i^- (resp. $b_{2\ell+1-i}^+$, $b_{2\ell+1-i}^-$), for $i = 1, \dots, \ell$, is isomorphic to M_ℓ. Since

$$\theta(b_i^\pm) = b_i^\pm, \qquad \theta(b_{2\ell+1-i}^\pm) = -b_{2\ell+1-i}^\pm,$$

for $i = 1, \dots, \ell$, we have

$$M_{2\ell}^0 \cong M_\ell \otimes M_\ell^0 \cong M_\ell \otimes L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0),$$

and for $\ell = 1$:

$$M_2^0 \cong M_1 \otimes M_1^0 \cong M_1 \otimes L_{A_1^{(1)}}(-\tfrac{1}{2}\Lambda_0).$$

5 Commutant of $L_{A_1^{(1)}}(-\Lambda_0)^{\otimes \ell}$ in $L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0) \otimes L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)$

In this section we use Weyl vertex algebra $M_{2\ell}$ to study certain subalgebras of $L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0) \otimes L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)$. We determine the commutants

$$\mathrm{Com}\big(L_{A_1^{(1)}}(-\Lambda_0)^{\otimes \ell}, L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0) \otimes L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0)\big)$$

and

$$\mathrm{Com}\big(L_{A_1^{(1)}}(-\Lambda_0), L_{A_1^{(1)}}(-\tfrac{1}{2}\Lambda_0) \otimes L_{A_1^{(1)}}(-\tfrac{1}{2}\Lambda_0)\big).$$

Let U be the vertex subalgebra of $M_\ell^0 \otimes M_\ell^0 \subset M_{2\ell}^0$ generated by

$$\begin{aligned} & e^{(i)} = a_i^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\mathbf{1}, \qquad f^{(i)} = a_i^-(-\tfrac{1}{2})a_{2\ell+1-i}^+(-\tfrac{1}{2})\mathbf{1} \qquad \text{and} \\ & h^{(i)} = (-a_i^+(-\tfrac{1}{2})a_i^-(-\tfrac{1}{2}) + a_{2\ell+1-i}^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2}))\mathbf{1}, \end{aligned} \tag{9}$$

$i = 1, \dots, \ell$. It is clear that U is isomorphic to the tensor product

$$\underbrace{L_{A_1^{(1)}}(-\Lambda_0) \otimes \cdots \otimes L_{A_1^{(1)}}(-\Lambda_0)}_{\ell \text{ times}}$$

of ℓ copies of the affine vertex algebra $L_{A_1^{(1)}}(-\Lambda_0)$.

Define also

$$H^{(i)} = (a_i^+(-\tfrac{1}{2})a_i^-(-\tfrac{1}{2}) + a_{2\ell+1-i}^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2}))\mathbf{1}, \qquad i = 1, \dots, \ell.$$

Then

$$H^{(i)} \in \mathrm{Com}(U, M_{2\ell}), \qquad i = 1, \dots, \ell.$$

Let

$$\mathfrak{h} = \bigoplus_{i=1}^{\ell} \mathbb{C} H^{(i)}.$$

Then $\mathfrak{h}$ can be considered as an Abelian Lie algebra, and the components of the vertex operators

$$Y(h, z) = \sum_{\mathfrak{n} \in \mathbb{Z}} h(n) z^{-n-1}, \qquad h \in \mathfrak{h},$$

define a representation of the associated Heisenberg algebra $\widehat{\mathfrak{h}}$. Moreover, $\mathfrak{h}$ generates the subalgebra of the $M_{2\ell}$ which is isomorphic to the Heisenberg vertex algebra $M_{\mathfrak{h}}(1)$ with central charge ℓ. We also note that $\langle H^{(i)}, H^{(j)} \rangle = H^{(i)}(1)H^{(j)} = -2\delta_{ij}$.

Using relations (9) one can show that the Virasoro vector (8) (in $M_{2\ell}$) can be written in a form:

$$\omega = \omega_1 + \omega_2$$

where

$$\omega_1 = \frac{1}{2} \sum_{i=1}^{\ell} \big(e^{(i)}(-1)f^{(i)}(-1) + f^{(i)}(-1)e^{(i)}(-1) + \tfrac{1}{2}h^{(i)}(-1)^2\big)\mathbf{1}$$

is the Virasoro vector in U and

$$\omega_2 = -\frac{1}{4}\sum_{i=1}^{\ell} H^{(i)}(-1)^2\mathbf{1}$$

is the Virasoro vector in $M_{\mathfrak{h}}(1)$. For $i=1,2$ let $Y(\omega_i,z)=\sum_{n\in\mathbb{Z}} L_i(n)z^{-n-2}$.

Proposition 1. *We have:*

$$\mathrm{Com}(U, M_{2\ell}) = M_{\mathfrak{h}}(1).$$

Proof. It is clear that $W = \mathrm{Com}(U, M_{2\ell})$ contains a subalgebra isomorphic to $M_{\mathfrak{h}}(1)$.

Assume now that $M_{\mathfrak{h}}(1) \neq W$. Then there is a vector $w \in W$, $\mathrm{wt}(w) > 0$ such that

$$H^{(i)}(n)w = \delta_{n,0}\lambda_{(i)}w, \qquad n\in\mathbb{Z}_{\geq 0}, \quad \lambda_{(i)}\in\mathbb{Z}, \quad i=1,\dots,\ell.$$

(Note that each $H^{(i)}(0)$ acts semisimply on $M_{2\ell}$ with eigenvalues in $\mathbb{Z}$.) By the definition of W we have that $L_1(0)w=0$. Therefore

$$L(0)w = L_2(0)w = -\frac{1}{4}\sum_{i=1}^{\ell}(\lambda_{(i)})^2 w.$$

This contradicts the fact that $\mathrm{wt}(w)>0$. So, $W = M_{\mathfrak{h}}(1)$. ■

Since $\theta(H^{(i)}) = -H^{(i)}$ for $i=1,\dots,\ell$ we have that $M_{\mathfrak{h}}(1)^+ = M_{\mathfrak{h}}(1)^0$ is a subalgebra of $M_\ell^0 \otimes M_\ell^0$. Therefore the vertex algebra $L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)\otimes L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)$ contains a subalgebra isomorphic to $U\otimes M_{\mathfrak{h}}(1)^+$. By using Proposition 1 we obtain the following theorem.

Theorem 2. *We have:*

$$\mathrm{Com}\big(L_{A_1^{(1)}}(-\Lambda_0)^{\otimes\ell}, L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0)\otimes L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0)\big) \cong M_{\mathfrak{h}}(1)^+.$$

In the case $\ell=1$, $\mathfrak{h}$ is a one-dimensional vector space $\mathfrak{h}=\mathbb{C}H$, where

$$H = \big(a_1^+(-\tfrac{1}{2})a_1^-(-\tfrac{1}{2}) + a_2^+(-\tfrac{1}{2})a_2^-(-\tfrac{1}{2})\big)\mathbf{1}.$$

Theorem 3. *We have:*

$$\mathrm{Com}\big(L_{A_1^{(1)}}(-\Lambda_0), L_{A_1^{(1)}}(-\tfrac{1}{2}\Lambda_0)\otimes L_{A_1^{(1)}}(-\tfrac{1}{2}\Lambda_0)\big) \cong M_{\mathfrak{h}}(1)^+.$$

In the next section we generalize Theorem 3 in another way.

6 Commutant of level -1 type $C_\ell^{(1)}$ affine vertex algebra in $L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)\otimes L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)$

In this section we study another subalgebra of $L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)\otimes L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)$. The subalgebra of $M_\ell^0\otimes M_\ell^0 \subset M_{2\ell}^0$ generated by

$$e_{2\epsilon_i}(=e^{(i)}) = a_i^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\mathbf{1}, \qquad f_{2\epsilon_i}(=f^{(i)}) = a_i^-(-\tfrac{1}{2})a_{2\ell+1-i}^+(-\tfrac{1}{2})\mathbf{1},$$
$$e_{\epsilon_i+\epsilon_j} = \tfrac{1}{2}\big(a_i^+(-\tfrac{1}{2})a_{2\ell+1-j}^-(-\tfrac{1}{2}) + a_j^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\big)\mathbf{1},$$
$$f_{\epsilon_i+\epsilon_j} = \tfrac{1}{2}\big(a_i^-(-\tfrac{1}{2})a_{2\ell+1-j}^+(-\tfrac{1}{2}) + a_j^-(-\tfrac{1}{2})a_{2\ell+1-i}^+(-\tfrac{1}{2})\big)\mathbf{1},$$

$$
\begin{aligned}
e_{\epsilon_i-\epsilon_j} &= \tfrac{1}{2}\big(a_i^+(-\tfrac{1}{2})a_j^-(-\tfrac{1}{2}) - a_{2\ell+1-j}^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\big)\mathbf{1},\\
f_{\epsilon_i-\epsilon_j} &= \tfrac{1}{2}\big(a_j^+(-\tfrac{1}{2})a_i^-(-\tfrac{1}{2}) - a_{2\ell+1-i}^+(-\tfrac{1}{2})a_{2\ell+1-j}^-(-\tfrac{1}{2})\big)\mathbf{1},\\
&\quad \text{for} \quad i,j=1,\dots,\ell, \quad i<j,
\end{aligned} \tag{10}
$$

is a level -1 affine vertex operator algebra associated to $C_\ell^{(1)}$. We denote it by $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$.

Let

$$
H = \sum_{i=1}^{\ell} H^{(i)} = \sum_{i=1}^{2\ell} a_i^+(-\tfrac{1}{2})a_i^-(-\tfrac{1}{2})\mathbf{1}.
$$

Set $\mathfrak{h}_1 = \mathbb{C}H \subset \mathfrak{h}$. Let $M_{\mathfrak{h}_1}(1)$ be the Heisenberg vertex algebra generated by H. Clearly $\langle H, H\rangle = -2\ell$.

Since $H \in \operatorname{Com}(\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0), M_{2\ell})$, we have that $M_{2\ell}$ contains a subalgebra isomorphic to $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0) \otimes M_{\mathfrak{h}_1}(1)$.

Proposition 2. *The Virasoro vector (8) in $M_{2\ell}$ can be written in a form:*

$$
\omega = \omega_1 + \omega_2,
$$

where ω_1 is the Virasoro vector (3) in $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$ obtained by the Sugawara construction and

$$
\omega_2 = -\frac{1}{4\ell} H(-1)^2\mathbf{1}
$$

is the Virasoro vector in $M_{\mathfrak{h}_1}(1)$.

Proof. Formula (3) implies that

$$
\begin{aligned}
\omega_1 = \frac{1}{2\ell}\Bigg(&\sum_{i=1}^{\ell}(e_{2\epsilon_i}(-1)f_{2\epsilon_i}(-1) + f_{2\epsilon_i}(-1)e_{2\epsilon_i}(-1))\mathbf{1}\\
&+ 2\sum_{\substack{i,j=1\\ i<j}}^{\ell}(e_{\epsilon_i+\epsilon_j}(-1)f_{\epsilon_i+\epsilon_j}(-1) + f_{\epsilon_i+\epsilon_j}(-1)e_{\epsilon_i+\epsilon_j}(-1))\mathbf{1}\\
&+ 2\sum_{\substack{i,j=1\\ i<j}}^{\ell}(e_{\epsilon_i-\epsilon_j}(-1)f_{\epsilon_i-\epsilon_j}(-1) + f_{\epsilon_i-\epsilon_j}(-1)e_{\epsilon_i-\epsilon_j}(-1))\mathbf{1} + \frac{1}{2}\sum_{i=1}^{\ell} h_{2\epsilon_i}(-1)^2\mathbf{1}\Bigg),
\end{aligned} \tag{11}
$$

where

$$
h_{2\epsilon_i}(= h^{(i)}) = \big(-a_i^+(-\tfrac{1}{2})a_i^-(-\tfrac{1}{2}) + a_{2\ell+1-i}^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\big)\mathbf{1}.
$$

It follows from relations (10) that

$$
\begin{aligned}
&(e_{2\epsilon_i}(-1)f_{2\epsilon_i}(-1) + f_{2\epsilon_i}(-1)e_{2\epsilon_i}(-1))\mathbf{1}\\
&\quad = 2a_i^+(-\tfrac{1}{2})a_i^-(-\tfrac{1}{2})a_{2\ell+1-i}^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\mathbf{1}\\
&\qquad + a_i^-(-\tfrac{3}{2})a_i^+(-\tfrac{1}{2})\mathbf{1} + a_{2\ell+1-i}^-(-\tfrac{3}{2})a_{2\ell+1-i}^+(-\tfrac{1}{2})\mathbf{1}\\
&\qquad - a_i^+(-\tfrac{3}{2})a_i^-(-\tfrac{1}{2})\mathbf{1} - a_{2\ell+1-i}^+(-\tfrac{3}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\mathbf{1},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&2(e_{\epsilon_i+\epsilon_j}(-1)f_{\epsilon_i+\epsilon_j}(-1) + f_{\epsilon_i+\epsilon_j}(-1)e_{\epsilon_i+\epsilon_j}(-1))\mathbf{1}\\
&\quad = a_i^+(-\tfrac{1}{2})a_i^-(-\tfrac{1}{2})a_{2\ell+1-j}^+(-\tfrac{1}{2})a_{2\ell+1-j}^-(-\tfrac{1}{2})\mathbf{1}
\end{aligned}
$$

$$
\begin{aligned}
&+ a_i^+(-\tfrac{1}{2})a_j^-(-\tfrac{1}{2})a_{2\ell+1-j}^-(-\tfrac{1}{2})a_{2\ell+1-i}^+(-\tfrac{1}{2})\mathbf{1}\\
&+ a_i^-(-\tfrac{1}{2})a_j^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})a_{2\ell+1-j}^+(-\tfrac{1}{2})\mathbf{1}\\
&+ a_j^+(-\tfrac{1}{2})a_j^-(-\tfrac{1}{2})a_{2\ell+1-i}^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\mathbf{1}\\
&+ \frac{1}{2}\Big(a_i^-(-\tfrac{3}{2})a_i^+(-\tfrac{1}{2})\mathbf{1} + a_j^-(-\tfrac{3}{2})a_j^+(-\tfrac{1}{2})\mathbf{1} + a_{2\ell+1-i}^-(-\tfrac{3}{2})a_{2\ell+1-i}^+(-\tfrac{1}{2})\mathbf{1}\\
&+ a_{2\ell+1-j}^-(-\tfrac{3}{2})a_{2\ell+1-j}^+(-\tfrac{1}{2})\mathbf{1} - a_i^+(-\tfrac{3}{2})a_i^-(-\tfrac{1}{2})\mathbf{1} - a_j^+(-\tfrac{3}{2})a_j^-(-\tfrac{1}{2})\mathbf{1}\\
&- a_{2\ell+1-i}^+(-\tfrac{3}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\mathbf{1} - a_{2\ell+1-j}^+(-\tfrac{3}{2})a_{2\ell+1-j}^-(-\tfrac{1}{2})\mathbf{1}\Big),
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
&2(e_{\epsilon_i-\epsilon_j}(-1)f_{\epsilon_i-\epsilon_j}(-1) + f_{\epsilon_i-\epsilon_j}(-1)e_{\epsilon_i-\epsilon_j}(-1))\mathbf{1}\\
&\quad = a_i^+(-\tfrac{1}{2})a_i^-(-\tfrac{1}{2})a_j^+(-\tfrac{1}{2})a_j^-(-\tfrac{1}{2})\mathbf{1}\\
&\quad - a_j^+(-\tfrac{1}{2})a_i^-(-\tfrac{1}{2})a_{2\ell+1-j}^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\mathbf{1}\\
&\quad - a_i^+(-\tfrac{1}{2})a_j^-(-\tfrac{1}{2})a_{2\ell+1-i}^+(-\tfrac{1}{2})a_{2\ell+1-j}^-(-\tfrac{1}{2})\mathbf{1}\\
&\quad + a_{2\ell+1-i}^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})a_{2\ell+1-j}^+(-\tfrac{1}{2})a_{2\ell+1-j}^-(-\tfrac{1}{2})\mathbf{1}\\
&\quad + \frac{1}{2}\Big(a_i^-(-\tfrac{3}{2})a_i^+(-\tfrac{1}{2})\mathbf{1} + a_j^-(-\tfrac{3}{2})a_j^+(-\tfrac{1}{2})\mathbf{1} + a_{2\ell+1-i}^-(-\tfrac{3}{2})a_{2\ell+1-i}^+(-\tfrac{1}{2})\mathbf{1}\\
&\quad + a_{2\ell+1-j}^-(-\tfrac{3}{2})a_{2\ell+1-j}^+(-\tfrac{1}{2})\mathbf{1} - a_i^+(-\tfrac{3}{2})a_i^-(-\tfrac{1}{2})\mathbf{1} - a_j^+(-\tfrac{3}{2})a_j^-(-\tfrac{1}{2})\mathbf{1}\\
&\quad - a_{2\ell+1-i}^+(-\tfrac{3}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2})\mathbf{1} - a_{2\ell+1-j}^+(-\tfrac{3}{2})a_{2\ell+1-j}^-(-\tfrac{1}{2})\mathbf{1}\Big),
\end{aligned}
\tag{14}
$$

for all $i, j = 1, \dots, \ell$, $i < j$. Using relations (11), (12), (13) and (14) one can obtain

$$
\omega_1 = \frac{1}{2}\sum_{i=1}^{2\ell}\left(a_i^-(-\tfrac{3}{2})a_i^+(-\tfrac{1}{2}) - a_i^+(-\tfrac{3}{2})a_i^-(-\tfrac{1}{2})\right)\mathbf{1} + \frac{1}{4\ell}H(-1)^2\mathbf{1},
$$

which implies the claim of proposition. ■

Using Proposition 2 and applying similar arguments as in the proof of Proposition 1, we obtain:

Proposition 3. *We have:*

$$
\mathrm{Com}\big(\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0), M_{2\ell}\big) = M_{\mathfrak{h}_1}(1).
$$

Since $\theta(H) = -H$, we have that $M_{\mathfrak{h}_1}(1)^+ = M_{\mathfrak{h}_1}(1)^0$ is a subalgebra of $M_\ell^0 \otimes M_\ell^0$. Therefore the vertex algebra $L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0) \otimes L_{C_\ell^{(1)}}(-\frac{1}{2}\Lambda_0)$ contains a subalgebra isomorphic to $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0) \otimes M_{\mathfrak{h}_1}(1)^+$. By using Proposition 3 we obtain the following theorem.

Theorem 4. *We have:*

$$
\mathrm{Com}\big(\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0), L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0) \otimes L_{C_\ell^{(1)}}(-\tfrac{1}{2}\Lambda_0)\big) \cong M_{\mathfrak{h}_1}(1)^+.
$$

7 The classification of ordinary modules for $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$

In this section we obtain a classification of irreducible $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$-modules, which we use in the following sections.

The following vertex algebra was considered in [1]: Let $\ell \geq 3$ and

$$V_{C_\ell^{(1)}}(-\Lambda_0) = \frac{N_{C_\ell^{(1)}}(-\Lambda_0)}{\langle \Delta_3(-1)\mathbf{1}\rangle},$$

where $\langle \Delta_3(-1)\mathbf{1}\rangle$ is the ideal generated by the singular vector $\Delta_3(-1)\mathbf{1}$, such that $\Delta_3(-1)$ is given by the following determinant:

$$\Delta_3(-1) = \begin{vmatrix} e_{2\epsilon_1}(-1) & e_{\epsilon_1+\epsilon_2}(-1) & e_{\epsilon_1+\epsilon_3}(-1) \\ e_{\epsilon_1+\epsilon_2}(-1) & e_{2\epsilon_2}(-1) & e_{\epsilon_2+\epsilon_3}(-1) \\ e_{\epsilon_1+\epsilon_3}(-1) & e_{\epsilon_2+\epsilon_3}(-1) & e_{2\epsilon_3}(-1) \end{vmatrix}.$$

Using relations (10), one can easily check that

$$\begin{vmatrix} e_{2\epsilon_1}(-1) & e_{\epsilon_1+\epsilon_2}(-1) & e_{\epsilon_1+\epsilon_3}(-1) \\ e_{\epsilon_1+\epsilon_2}(-1) & e_{2\epsilon_2}(-1) & e_{\epsilon_2+\epsilon_3}(-1) \\ e_{\epsilon_1+\epsilon_3}(-1) & e_{\epsilon_2+\epsilon_3}(-1) & e_{2\epsilon_3}(-1) \end{vmatrix} \mathbf{1} = 0$$

in $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$, so $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$ is a certain quotient of $V_{C_\ell^{(1)}}(-\Lambda_0)$, for $\ell \geq 3$. Thus, any irreducible module for $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$ is an irreducible module for $V_{C_\ell^{(1)}}(-\Lambda_0)$. The classification of all irreducible modules in the category $\mathcal{O}$ for $V_{C_3^{(1)}}(-\Lambda_0)$ was obtained in [1, Example 4.1]. We will apply this result to obtain a classification of all irreducible ordinary $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$-modules. (Recall that a module is called ordinary if $L(0)$ acts semisimply with finite-dimensional weight spaces.)

Proposition 4. *Let $\ell \geq 3$. The set*

$$\{L_{C_\ell^{(1)}}((-n-1)\Lambda_0 + n\Lambda_1) \mid n \in \mathbb{Z}_{\geq 0}\} \cup \{L_{C_\ell^{(1)}}(-2\Lambda_0 + \Lambda_2)\} \tag{15}$$

provides a complete list of irreducible ordinary modules for the vertex operator algebras $V_{C_\ell^{(1)}}(-\Lambda_0)$ and $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$.

Proof. We use the well-known method for classification of highest weights of $V_{C_\ell^{(1)}}(-\Lambda_0)$-modules as solutions of certain polynomial equations arising from the singular vectors (cf. [2, 5, 8, 29, 30, 31]). The highest weights of ordinary modules are of the form $-\Lambda_0 + \mu$, where $\mu = \sum_{i=1}^{\ell} h_i \epsilon_i$. Clearly, $h_i \in \mathbb{Z}_{\geq 0}$, for $i = 1, \ldots, \ell$. Using polynomials from [1, Example 4.1], and the fact that

$$\tfrac{1}{2} f_{\epsilon_3-\epsilon_i}(0)^2 \Delta_3(-1)\mathbf{1} = \begin{vmatrix} e_{2\epsilon_1}(-1) & e_{\epsilon_1+\epsilon_2}(-1) & e_{\epsilon_1+\epsilon_i}(-1) \\ e_{\epsilon_1+\epsilon_2}(-1) & e_{2\epsilon_2}(-1) & e_{\epsilon_2+\epsilon_i}(-1) \\ e_{\epsilon_1+\epsilon_i}(-1) & e_{\epsilon_2+\epsilon_i}(-1) & e_{2\epsilon_i}(-1) \end{vmatrix} \mathbf{1},$$

for $i = 4, \ldots, \ell$, we obtain that the weights μ are annihilated by the polynomials

$$\begin{aligned} p_i(\mu) &= (h_1+1)(h_2+\tfrac{1}{2})h_i, \\ q_i(\mu) &= (h_1+1)(4h_i + (h_2+h_i)(h_2+h_i-1)), \\ r_i(\mu) &= 4h_i(h_2+1) + (h_1+h_i-1)(h_2+h_i+h_2(h_1+h_i)), \end{aligned}$$

for $i = 3, \ldots, \ell$. Then one easily obtains that $h_i = 0$, for $i = 3, \ldots, \ell$, and that either $h_2 = 0$ and $h_1 = n$, for $n \in \mathbb{Z}_{\geq 0}$ or $h_2 = 1$ and $h_1 = 1$. So we have proved that any irreducible ordinary $V_{C_\ell^{(1)}}(-\Lambda_0)$-module (resp. $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$-module) must belong to the set (15).

Since $a_1^+(-1/2)^n\mathbf{1} \in M_{2\ell}$ is a singular vector of highest weight $-(n+1)\Lambda_0 + n\Lambda_1$, and

$$e^*_{\epsilon_1+\epsilon_2} = \tfrac{1}{2}(a_1^+(-\tfrac{1}{2})a^-_{2\ell-1}(-\tfrac{1}{2})\mathbf{1} - a_2^+(-\tfrac{1}{2})a^-_{2\ell}(-\tfrac{1}{2})\mathbf{1}) \in M_{2\ell}$$

is a singular vector of highest weight $-2\Lambda_0 + \Lambda_2$, we have that every module from the set (15) is a module for these vertex operator algebras. The proof is now complete. ■

Remark 2. In our new paper [6] we prove that the vertex operator algebra $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$ is simple. Therefore, Proposition 4 also gives the classification of irreducible $L_{C_\ell^{(1)}}(-\Lambda_0)$-modules.

8 Conformal embedding of $C_\ell^{(1)}$ into $A_{2\ell-1}^{(1)}$ at level -1

In this section we show that $L_{C_\ell^{(1)}}(-\Lambda_0)$ is a $\mathbb{Z}_2$-orbifold of vertex operator algebra $L_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$, and determine the corresponding decomposition.

The subalgebra of $M_{2\ell}$ generated by

$$e^A_{\epsilon_i-\epsilon_j} = a_i^+(-\tfrac{1}{2})a_j^-(-\tfrac{1}{2})\mathbf{1}, \qquad f^A_{\epsilon_i-\epsilon_j} = a_i^-(-\tfrac{1}{2})a_j^+(-\tfrac{1}{2})\mathbf{1} \qquad \text{for } i,j = 1,\dots,2\ell, \quad i<j,$$

is a level -1 affine vertex operator algebra associated to the affine Lie algebra $\hat{\mathfrak{g}}_1$ of type $A_{2\ell-1}^{(1)}$. We denote it by $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$. Clearly, $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$ is a subalgebra of $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$. Let $\hat{\mathfrak{g}}$ be the affine Lie algebra of type $C_\ell^{(1)}$.

As before, denote by ω_1 the Virasoro vector in $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$, and by ω_1^A the Virasoro vector in $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$.

Proposition 5. *We have*

$$\omega_1 = \omega_1^A.$$

Proof. Similarly as in Proposition 2, one can show that

$$\omega_1^A = \frac{1}{2}\sum_{i=1}^{2\ell}\left(a_i^-(-\tfrac{3}{2})a_i^+(-\tfrac{1}{2}) - a_i^+(-\tfrac{3}{2})a_i^-(-\tfrac{1}{2})\right)\mathbf{1} + \frac{1}{4\ell}H(-1)^2\mathbf{1},$$

which implies the claim of proposition. ■

Furthermore, the vector $e^*_{\epsilon_1+\epsilon_2}$ from the proof of Proposition 4 is a singular vector for $\hat{\mathfrak{g}}$ in $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$ which generates $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$-module $\widetilde{L}_{C_\ell^{(1)}}(-2\Lambda_0+\Lambda_2)$, whose top component is irreducible $\mathfrak{g}$-module $V_{C_\ell}(\omega_2)$. Clearly, θ acts as 1 on $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$ and as -1 on $\widetilde{L}_{C_\ell^{(1)}}(-2\Lambda_0+\Lambda_2)$.

Lemma 1. *Let $u,v \in \widetilde{L}_{C_\ell^{(1)}}(-2\Lambda_0+\Lambda_2)$. Then $u_n v \in \widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$, for any $n \in \mathbb{Z}$.*

Proof. It suffices to prove the lemma for u and v from top component $R(0)$ of $\widetilde{L}_{C_\ell^{(1)}}(-2\Lambda_0+\Lambda_2)$. Then the statement will follow from the associator formulae.

First we notice that

$$u_0 v \in \widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0),$$

(since vectors of conformal weight 1 with bracket $[u,v] = u_0 v$ span Lie algebra $\mathfrak{g}_1$ of type $A_{2\ell-1}$, and fixed point subalgebra $\mathfrak{g}$ is a Lie algebra of type C_ℓ).

Assume now that

$$u_{n_0}v \notin \widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$$

for certain $u, v \in R(0)$ and $n_0 \in \mathbb{Z}$. Take maximal n_0 with this property.

Then $u_{n_0}v$ has nontrivial component in some highest weight $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$-module W of highest weight $-\Lambda_0 + \mu$, and therefore there is a nontrivial intertwining operator of type

$$\binom{W}{\widetilde{L}_{C_\ell^{(1)}}(-2\Lambda_0 + \Lambda_2) \quad \widetilde{L}_{C_\ell^{(1)}}(-2\Lambda_0 + \Lambda_2)}.$$

One can associate to this intertwining operator, a non-trivial $\mathfrak{g}$-homomorphism

$$f: \ V_{C_\ell}(\omega_2) \otimes V_{C_\ell}(\omega_2) \to V_{C_\ell}(\mu).$$

In particular, $V_{C_\ell}(\mu)$ must appear in the decomposition of tensor product $V_{C_\ell}(\omega_2) \otimes V_{C_\ell}(\omega_2)$.

First consider the case $\ell = 2$. Using the decomposition of tensor product $V_{C_2}(\omega_2) \otimes V_{C_2}(\omega_2)$ from (7) and the fact that the lowest conformal weights of modules of highest weights $-\Lambda_0 + 2\omega_2$ and $-\Lambda_0 + 2\omega_1$ are $\frac{5}{2}$ and $\frac{3}{2}$, respectively, we conclude that these modules cannot appear inside $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$. Thus, $u_{n_0}v \in \widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$.

Now, let $\ell \geq 3$. Relation (7) implies that the only cases of weights $-\Lambda_0 + \mu$ (aside from $\mu = 0$) from Proposition 4 such that μ appears in the decomposition of $V_{C_\ell}(\omega_2) \otimes V_{C_\ell}(\omega_2)$ are when $\mu = 2\omega_1$ or $\mu = \omega_2$. Since $u_{n_0}v$ is θ-invariant, it does not contain component inside $\widetilde{L}_{C_\ell^{(1)}}(-2\Lambda_0 + \Lambda_2)$. On the other hand, the lowest conformal weight of module of highest weight $-\Lambda_0 + 2\omega_1$ is $\frac{\ell+1}{\ell}$, which is not an integer. Thus, this module cannot appear inside $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$. Thus, $u_{n_0}v \in \widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0)$. ■

Theorem 5. *We have:*

$$\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0) = \widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0) \oplus \widetilde{L}_{C_\ell^{(1)}}(-2\Lambda_0 + \Lambda_2).$$

In particular,

$$\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0) = \widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)^0, \qquad \widetilde{L}_{C_\ell^{(1)}}(-2\Lambda_0 + \Lambda_2) = \widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)^1.$$

Proof. Lemma 1 shows that $\widetilde{L}_{C_\ell^{(1)}}(-\Lambda_0) \oplus \widetilde{L}_{C_\ell^{(1)}}(-2\Lambda_0 + \Lambda_2)$ is a vertex subalgebra of $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$. But this subalgebra clearly contains generators of $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$, which implies the claim of theorem. ■

The classification of irreducible ordinary $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$-modules follows from the results from [8] and similar arguments as in Section 7:

Proposition 6. *The set*

$$\left\{L_{A_{2\ell-1}^{(1)}}((-n-1)\Lambda_0 + n\Lambda_1) \mid n \in \mathbb{Z}_{\geq 0}\right\} \cup \left\{L_{A_{2\ell-1}^{(1)}}((-n-1)\Lambda_0 + n\Lambda_{2\ell-1}) \mid n \in \mathbb{Z}_{\geq 0}\right\}$$

provides a complete list of irreducible ordinary modules for the vertex operator algebra $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$.

The following result shows that most irreducible $\widetilde{L}_{A^{(1)}_{2\ell-1}}(-\Lambda_0)$-modules remain irreducible when we restrict them on $\widetilde{L}_{C^{(1)}_{\ell}}(-\Lambda_0)$.

Proposition 7. *Assume that $\ell \geq 3$, $n \in \mathbb{Z}_{>0}$. Then we have the following isomorphisms of $\widetilde{L}_{C^{(1)}_{\ell}}(-\Lambda_0)$-modules:*

$$L_{A^{(1)}_{2\ell-1}}(-(n+1)\Lambda_0 + n\Lambda_1) \cong L_{C^{(1)}_{\ell}}(-(n+1)\Lambda_0 + n\Lambda_1),$$
$$L_{A^{(1)}_{2\ell-1}}(-(n+1)\Lambda_0 + n\Lambda_{2\ell-1}) \cong L_{C^{(1)}_{\ell}}(-(n+1)\Lambda_0 + n\Lambda_1).$$

Proof. We use the Theorem 6.1 from [14]. The definition of automorphism θ then implies (in the notation of [14]) that

$$\theta \circ L_{A^{(1)}_{2\ell-1}}(-(n+1)\Lambda_0 + n\Lambda_1) \cong L_{A^{(1)}_{2\ell-1}}(-(n+1)\Lambda_0 + n\Lambda_{2\ell-1}),$$

as $\widetilde{L}_{A^{(1)}_{2\ell-1}}(-\Lambda_0)$-modules.

Theorem 6.1 from [14] now implies that $L_{A^{(1)}_{2\ell-1}}(-(n+1)\Lambda_0 + n\Lambda_1)$ and $L_{A^{(1)}_{2\ell-1}}(-(n+1)\Lambda_0 + n\Lambda_{2\ell-1})$ are irreducible as $\widetilde{L}_{C^{(1)}_{\ell}}(-\Lambda_0)$-modules. The claim of Proposition now follows easily. ■

In [6] we shall prove that $\widetilde{L}_{A^{(1)}_{2\ell-1}}(-\Lambda_0)$ and $\widetilde{L}_{C^{(1)}_{\ell}}(-\Lambda_0)$ are simple. We don't use this result in the present paper. But even without these simplicity results we can conclude that the analogous of Theorem 5 also holds for simple vertex operator algebras $L_{A^{(1)}_{2\ell-1}}(-\Lambda_0)$ and $L_{C^{(1)}_{\ell}}(-\Lambda_0)$.

Corollary 1. *We have:*

$$L_{A^{(1)}_{2\ell-1}}(-\Lambda_0) = L_{C^{(1)}_{\ell}}(-\Lambda_0) \oplus L_{C^{(1)}_{\ell}}(-2\Lambda_0 + \Lambda_2).$$

Proof. First we notice that the automorphism θ also naturally acts on a simple vertex operator algebra $L_{A^{(1)}_{2\ell-1}}(-\Lambda_0)$. Theorem 5 implies that

$$L_{A^{(1)}_{2\ell-1}}(-\Lambda_0) = V^0 \oplus V^1$$

where V^0 (resp. V^1) is a quotient of $\widetilde{L}_{C^{(1)}_{\ell}}(-\Lambda_0)$ (resp. $\widetilde{L}_{C^{(1)}_{\ell}}(-2\Lambda_0 + \Lambda_2)$). By using the fact that $\mathbb{Z}_2$-orbifold components of simple vertex operator algebra are simple (cf. [14]), we get that $V^0 = L_{C^{(1)}_{\ell}}(-\Lambda_0)$ and $V^1 = L_{C^{(1)}_{\ell}}(-2\Lambda_0 + \Lambda_2)$. The proof follows. ■

9 Commutant of $L_{A^{(1)}_1}(-\Lambda_0)^{\otimes \ell}$ in $L_{C^{(1)}_{\ell}}(-\Lambda_0)$

We shall now study vertex operator algebra $L_{C^{(1)}_{\ell}}(-\Lambda_0)$. Corollary 1 implies that $L_{C^{(1)}_{\ell}}(-\Lambda_0)$ is a $\mathbb{Z}_2$-orbifold of vertex operator algebra $L_{A^{(1)}_{2\ell-1}}(-\Lambda_0)$. Clearly, $L_{A^{(1)}_{2\ell-1}}(-\Lambda_0)$ is a quotient of $\widetilde{L}_{A^{(1)}_{2\ell-1}}(-\Lambda_0)$ modulo the maximal submodule of $\widetilde{L}_{A^{(1)}_{2\ell-1}}(-\Lambda_0)$ (which is possibly zero).

As before, we denote

$$H^{(i)} = (a_i^+(-\tfrac{1}{2})a_i^-(-\tfrac{1}{2}) + a_{2\ell+1-i}^+(-\tfrac{1}{2})a_{2\ell+1-i}^-(-\tfrac{1}{2}))\mathbf{1}, \quad i = 1, \dots, \ell,$$
$$H = H^{(1)} + \cdots + H^{(\ell)}.$$

Also, let

$$\overline{H}^{(i)} = H^{(i)} - H^{(i+1)}, \quad i = 1, \dots, \ell - 1, \qquad \text{and}$$
$$\overline{\mathfrak{h}}_1 = \mathbb{C}\overline{H}^{(1)} + \cdots + \mathbb{C}\overline{H}^{(\ell-1)}.$$

Clearly $\langle \overline{H}^{(i)}, \overline{H}^{(j)} \rangle = -4\delta_{i,j}$.

Theorem 6. *We have:*

$$\mathrm{Com}\big(L_{A_1^{(1)}}(-\Lambda_0)^{\otimes \ell}, L_{C_\ell^{(1)}}(-\Lambda_0)\big) \cong M_{\overline{\mathfrak{h}}_1}(1)^+.$$

Proof. By using same arguments as in the proof of Proposition 1 we get that:

$$\mathrm{Com}\big(L_{A_1^{(1)}}(-\Lambda_0)^{\otimes \ell}, \widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)\big) = M_{\overline{\mathfrak{h}}_1}(1).$$

Since generators $\overline{H}^{(i)}$ don't belong to the maximal submodule of $\widetilde{L}_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$ (which is possibly zero) we conclude that

$$\mathrm{Com}\big(L_{A_1^{(1)}}(-\Lambda_0)^{\otimes \ell}, L_{A_{2\ell-1}^{(1)}}(-\Lambda_0)\big) \cong M_{\overline{\mathfrak{h}}_1}(1). \tag{16}$$

Let θ be the automorphism of order two of the vertex operator algebra $L_{A_{2\ell-1}^{(1)}}(-\Lambda_0)$ as in Corollary 1. Then

$$\theta(\overline{H}^{(i)}) = -\overline{H}^{(i)}, \qquad i = 1, \dots, \ell - 1,$$

and we get that

$$M_{\overline{\mathfrak{h}}_1}(1)^+ \subset L_{A_{2\ell-1}^{(1)}}(-\Lambda_0)^0 = L_{C_\ell^{(1)}}(-\Lambda_0). \tag{17}$$

Now proof of the theorem follows from relations (16) and (17). ■

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