ON CONVERGENCE OF AN ITERATIVE PROCESS WITH ERRORS FOR A
FINITE FAMILY OF MULTI-VALUED MAPPINGS

NAZLI KADIOGLU AND ISA YILDIRIM

Abstract. In this paper, we introduce an iterative process with errors for a finite family of multi-valued mappings satisfying the condition \((C)\) which is weaker than nonexpansiveness and prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

1. Introduction and Preliminaries

Let \(E\) be a nonempty and convex subset of a Banach space \(X\). The set \(E\) is called proximal if for each \(x \in X\), there exists an element \(y \in E\) such that

\[
\|x - y\| = \text{dist}(x, E) = \inf \{\|x - z\| : z \in E\}.
\]

It is known that every closed convex subset of a uniformly convex Banach space is proximal. We denote by \(CB(E), K(E)\) and \(P(E)\) the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of \(E\) respectively. The Hausdorff metric \(H\) on \(CB(X)\) is defined by

\[
H(A, B) = \max \{\sup_{x \in A} \text{dist}(x, B), \sup_{y \in B} \text{dist}(y, A)\}
\]

for all \(A, B \in CB(X)\). Let \(T : X \to 2^X\) be a multi-valued mapping. An element \(x \in X\) is said to be a fixed point of \(T\), if \(x \in Tx\). The set of fixed points of \(T\) will be denote by \(F(T)\).

**Definition 1.** A multi-valued mapping \(T : X \to CB(X)\) is called

(i) nonexpansive if \(H(Tx, Ty) \leq \|x - y\|\), for all \(x, y \in X\);
(ii) quasi-nonexpansive if \(F(T) \neq \emptyset\) and \(H(Tx, Tp) \leq \|x - p\|\), for all \(x \in X\) and all \(p \in F(T)\);
(iii) hemi-compact if, for any sequence \(\{x_n\}\) in \(X\) such that \(\text{dist}(x_n, Tx_n) \to 0\) as \(n \to \infty\), there exists a subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) such that \(x_{n_j} \to p \in X\). We note that if \(X\) is compact, then every multi-valued mappings \(T : X \to CB(X)\) is hemi-compact.

It is clear that every multi-valued nonexpansive mapping \(T\) with \(F(T) \neq \emptyset\) is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive (see [13]). It is known that if \(T\) is a multi-valued quasi-nonexpansive mapping, then \(F(T)\) is closed.

In 2008, Suzuki [14] introduced a condition on mappings, called \((C)\) which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Very recently, Abkar and Eslamian [1] used a modified Suzuki condition for multi-valued mappings as follows:

**Definition 2.** A multi-valued mapping \(T : X \to CB(X)\) is said to satisfy condition \((C)\) provided that

\[
\frac{1}{2} \text{dist}(x, Tx) \leq \|x - y\| \Rightarrow H(Tx, Ty) \leq \|x - y\|
\]

for all \(x, y \in X\).

**Lemma 1.** [1] Let \(T : X \to CB(X)\) be a multi-valued nonexpansive mapping, then \(T\) satisfies the condition \((C)\).

2000 Mathematics Subject Classification. 47H10, 47H09.
Key words and phrases. Multi-valued Mapping, Common Fixed Point, Condition \((C)\), Strong and Weak Convergence.
Lemma 2. [5] Let \( T : X \rightarrow CB(X) \) be a multi-valued mapping which satisfies the condition (C) and has a fixed point. Then \( T \) is a quasi-nonexpansive mapping.

Lemma 3. [5] Let \( E \) be a nonempty subset of a Banach space \( X \). Suppose \( T : E \rightarrow P(E) \) satisfies condition (C) then
\[
H(Tx, Ty) \leq 2\text{dist}(x, Tx) + \|x - y\|
\]
holds for all \( x, y \in E \).

The study of fixed points for multi-valued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [7] and Nadler [8]. Since then the theory of multi-valued mappings has applications in control theory, convex optimization, differential equations and economics. Theory of multi-valued nonexpansive mappings is harder than the corresponding theory of single-valued nonexpansive mappings. Different iterative processes have been used to approximate fixed points of multi-valued nonexpansive mappings.

Among these iterative processes, Sastry and Babu [11] considered the following.

Let \( E \) be a nonempty convex subset of a Banach space \( X \), \( T : E \rightarrow P(E) \) a multi-valued mapping with \( p \in Tp \).

(i) The sequence of Mann iterates is defined by \( x_1 \in E \),
\[
x_{n+1} = (1 - a_n)x_n + a_n y_n, \quad n \geq 1,
\]
where \( y_n \in Tx_n \);

(ii) The sequence of Ishikawa iterates is defined by \( x_1 \in E \),
\[
\begin{align*}
y_n &= (1 - \beta_n)x_n + \beta_n z_n, \quad n \geq 1, \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n u_n, \quad n \geq 1,
\end{align*}
\]
where \( z_n \in Tx_n \) and \( u_n \in Ty_n \).

They proved that the Mann and Ishikawa iteration processes for multi-valued mapping \( T \) with a fixed point \( p \) converge to a fixed point \( q \) of \( T \) under certain conditions. They also claimed that the fixed point \( q \) may be different from \( p \). Panyanak [10] extended result of Sastry and Babu [11] to uniformly convex Banach spaces. After, Song and Wang [12] noted that there was a gap in the proof of the main result in [10]. They further revised the gap and also gave the affirmative answer to Panyanak’s open question. Shahzad and Zegeye [13] extended and improved results already appeared in the papers [10, 11, 12]. Recently, Cholamjiak and Suantai [3] introduced the following Ishikawa iteration with errors for two multi-valued quasi-nonexpansive mappings and prove some convergence theorems for such mappings.

Let \( E \) be a nonempty convex subset of a Banach space \( X \) and \( T_1, T_2 : E \rightarrow CB(E) \) be two multi-valued mappings. Then for \( x_1 \in E \),
\[
\begin{align*}
y_n &= (1 - a_n - b_n)x_n + a_n w_n + b_n u_n, \quad n \geq 1, \\
x_{n+1} &= (1 - \alpha_n - \beta_n)x_n + \alpha_n z_n + \beta_n v_n, \quad n \geq 1,
\end{align*}
\]
where \( w_n \in T_2x_n \) and \( z_n \in T_1y_n \), \( \{a_n\}, \{b_n\}, \{\alpha_n\}, \{\beta_n\} \in [0, 1] \) and \( \{u_n\}, \{v_n\} \) are bounded sequences in \( E \).

Very recently, Eslamian and Homaeipour [6] introduced a new three-step iterative process for multi-valued mappings in Banach spaces. And they proved some convergence theorems for multi-valued mappings satisfying condition (C) in uniformly convex Banach spaces. Their iteration process is a generalization of Noor iteration process with errors as follows:

Let \( E \) be a nonempty convex subset of a Banach space \( X \) and \( T_1, T_2, T_3 : E \rightarrow CB(E) \) be three multi-valued mappings. Then for \( x_1 \in E \),
\[
\begin{align*}
w_n &= (1 - a_n - b_n)x_n + a_n z_n + b_n s_n, \quad n \geq 1, \\
y_n &= (1 - c_n - d_n)x_n + c_n u_n + d_n s_n, \quad n \geq 1, \\
x_{n+1} &= (1 - \alpha_n - \beta_n)x_n + \alpha_n v_n + \beta_n s_n, \quad n \geq 1,
\end{align*}
\]
where \( z_n \in T_1 x_n, u_n \in T_2 w_n \) and \( v_n \in T_3 y_n \), \( \{a_n\}, \{b_n\}, \{c_n\}, \{d_n\}, \{\alpha_n\}, \{\beta_n\} \in [0, 1] \) and \( \{s_n\}, \{s'_n\} \) and \( \{s''_n\} \) are bounded sequences in \( E \).

Finding common fixed points of a finite family \( \{T_i : i = 1, 2, ..., k\} \) of mappings acting on a Hilbert space is a problem that often arises in applied mathematics. In fact, many algorithms have been introduced for different classes of mappings with a nonempty set of common fixed points. Unfortunately, the existence results of common fixed points of a family of mappings are not known in many situations. Therefore, it is natural to consider approximation results for these classes of mappings.

2. Main Results

In this section, we use the following iteration process.

(A): Let \( E \) be a nonempty convex subset of a Banach space \( X \) and \( T_i : E \rightarrow CB(E) \) \( (i = 1, 2, ..., k) \) be a finite family of multi-valued mappings. Then for \( x_1 \in E \),

\[
\begin{align*}
y_{1n} &= (1 - \alpha_{1n} - \beta_{1n}) x_n + \alpha_{1n} z_{n,1} + \beta_{1n} u_{1n}, \quad n \geq 1, \\
y_{2n} &= (1 - \alpha_{2n} - \beta_{2n}) x_n + \alpha_{2n} z_{n,2} + \beta_{2n} u_{2n}, \quad n \geq 1, \\
&\vdots \\
y_{(k-1)n} &= (1 - \alpha_{(k-1)n} - \beta_{(k-1)n}) x_n + \alpha_{(k-1)n} z_{n,k-1} + \beta_{(k-1)n} u_{(k-1)n}, \quad n \geq 1, \\
y_{n+1} &= (1 - \alpha_{kn} - \beta_{kn}) x_n + \alpha_{kn} z_{n,k} + \beta_{kn} u_{kn}, \quad n \geq 1,
\end{align*}
\]

where \( z_{n,1} \in T_1 (x_n) \) and \( z_{n,i} \in T_i (y_{(i-1)n}) \) for \( i = 2, 3, ..., k \) and \( \{\alpha_n\}, \{\beta_n\} \in [0, 1] \) and \( \{u_n\} \) are bounded sequences in \( E \).

Clearly, this iteration process generalizes the Mann iteration (1), the Ishikawa iteration (2), the Ishikawa iteration with errors (3) and the three-step iteration process (4) from one mapping to the finite family of mappings \( \{T_i : i = 1, 2, ..., k\} \).

**Definition 3.** A mapping \( T : E \rightarrow CB(E) \) is said to satisfy condition (I) if there is a nondecreasing function \( g : [0, \infty) \rightarrow [0, \infty) \) with \( g(0) = 0 \), \( g(r) > 0 \) for all \( r \in (0, \infty) \) such that

\[
dist (x, Tx) \geq g (\dist (x, F(T))).
\]

Let \( T_i : E \rightarrow CB(E) \) \( (i = 1, 2, ..., k) \) be a finite family of mappings. The mappings \( T_i \) for all \( i \) \( (i = 1, 2, ..., k) \) are said to satisfy condition (II) if there exist a nondecreasing function \( g : [0, \infty) \rightarrow [0, \infty) \) with \( g(0) = 0 \), \( g(r) > 0 \) for all \( r \in (0, \infty) \) such that

\[
\sum_{i=1}^{k} \dist (x, T_ix) \geq g (\dist (x, F)),
\]

where \( F = \cap_{i=1}^{k} F(T_i) \).

Throughout this paper, we denote the weak convergence and the strong convergence by \( \rightharpoonup \) and \( \rightarrow \), respectively.

**Definition 4.** A Banach space \( E \) is said to satisfy Opial’s condition [9] if for any sequence \( \{x_n\} \) in \( E \), \( x_n \rightharpoonup x \) implies that

\[
\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|
\]

for all \( y \in E \) with \( y \neq x \).

Examples of Banach spaces satisfying this condition are Hilbert spaces and all \( L^p \) spaces \( (1 < p < \infty) \). On the other hand, \( L^p[0, 2\pi] \) with \( 1 < p \neq 2 \) fail to satisfy Opial’s condition.

The mapping \( T : E \rightarrow CB(E) \) is called demi-closed if for every sequence \( \{x_n\} \subset E \) and any \( y_n \in Tx_n \) such that \( x_n \rightharpoonup x \) and \( y_n \rightarrow y \), we have \( x \in E \) and \( y \in Tx \).

**Remark 1.** [4] If the space \( E \) satisfies Opial’s condition, then \( I - T \) is demi-closed at 0, where \( T : E \rightarrow K(E) \) is a multi-valued nonexpansive mapping.

We use the following lemmas to prove our main results.
Lemma 4. [15] Let \( \{a_n\}, \{b_n\} \) and \( \{\delta_n\} \) be sequences of nonnegative real numbers satisfying the inequality
\[
a_{n+1} \leq (1 + \delta_n) a_n + b_n.
\]
If \( \sum_{n=1}^{\infty} \delta_n < \infty \) and \( \sum_{n=1}^{\infty} b_n < \infty \), then \( \lim_{n \to \infty} a_n \) exists. In particular, if \( \{a_n\} \) has a subsequence converging to 0, then \( \lim_{n \to \infty} a_n = 0 \).

Lemma 5. [2] Let \( X \) be a uniformly convex Banach space and \( B_r = \{ x \in X : \|x\| \leq r \}, \ r > 0 \). Then there exists a continuous, strictly increasing and convex function \( \varphi : [0, \infty) \to [0, \infty) \) with \( \varphi(0) = 0 \) such that
\[
\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha \beta \varphi(\|x - y\|),
\]
for all \( x, y, z \in B_r \) and all \( \alpha, \beta, \gamma \in [0, 1] \) with \( \alpha + \beta + \gamma = 1 \).

Theorem 6. Let \( E \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \). Let \( T_i : E \to CB(E), (i = 1, 2, ..., k) \) be a finite family of multi-valued mappings satisfying the condition (C). Assume that \( F = \cap_{i=1}^{k} F(T_i) \neq \emptyset \) and \( T_i(p) = \{x\}, (i = 1, 2, ..., k) \) for each \( p \in F \). Let \( \{x_n\} \) be the iterative process defined by (A), and \( \alpha_i + \beta_i \in [a, b] \subset (0, 1) \) for \( i = 1, 2, ..., k \) and \( \sum_{n=1}^{\infty} \beta_i < \infty \) for each \( i \). Assume that \( T_i (i = 1, 2, ..., k) \) satisfying the condition (II). Then \( \{x_n\} \) converges strongly to a common fixed point of \( T_i \) for \( i = 1, 2, ..., k \).

Proof. We split the proof into three steps.

Step 1. Show that \( \lim_{n \to \infty} \|x_n - p\| \) exists for any \( p \in F \).

Let \( p \in F \). Since the sequences \( \{u_{in}\} \) are bounded for \( i = 1, 2, ..., k \), there exists \( M > 0 \) such that
\[
\max \left\{ \sup_{n \geq 1} \|u_{1n} - p\|, \sup_{n \geq 1} \|u_{2n} - p\|, ..., \sup_{n \geq 1} \|u_{kn} - p\| \right\} \leq M.
\]
Using (A) and quasi-nonexpansiveness of \( T_i (i = 1, 2, ..., k) \) we have
\[
\|y_{1n} - p\| \leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} \|z_{n,1} - p\| + \beta_{1n} \|u_{1n} - p\|
\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} \text{dist}(z_{n,1}, T_1(p)) + \beta_{1n} \|u_{1n} - p\|
\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} H(T_1(x_n), T_1(p)) + \beta_{1n} \|u_{1n} - p\|
\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} \|x_n - p\| + \beta_{1n} \|u_{1n} - p\|
= (1 - \beta_{1n}) \|x_n - p\| + \alpha_{1n} \|x_n - p\| + \beta_{1n} \|u_{1n} - p\|
\leq \|x_n - p\| + \beta_{1n} M
\]
and
\[
\|y_{2n} - p\| \leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \|z_{n,2} - p\| + \beta_{2n} \|u_{2n} - p\|
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \text{dist}(z_{n,2}, T_2(p)) + \beta_{2n} \|u_{2n} - p\|
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} H(T_2(y_{1n}), T_2(p)) + \beta_{2n} \|u_{2n} - p\|
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \|y_{1n} - p\| + \beta_{2n} \|u_{2n} - p\|
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \|x_n - p\| + \beta_{2n} \|u_{2n} - p\|
\leq (1 - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \beta_{1n} M + \beta_{2n} M
\leq \|x_n - p\| + (\beta_{1n} + \beta_{2n}) M.
\]
Similarly, we have
\[
\|y_{(k-1)n} - p\| \leq \|x_n - p\| + (\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n}) M.
\]
and also
\[
\|x_{n+1} - p\| \leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\| + \alpha_{kn} \|z_{n,i} - p\| + \beta_{kn} \|u_{kn} - p\|
\]
\[
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\| + \alpha_{kn} \text{dist}(z_{n,k}, T_k(p)) + \beta_{kn} \|u_{kn} - p\|
\]
\[
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\| + \alpha_{kn} H(T_k(y_{k(1)n}) , T_k(p)) + \beta_{kn} \|u_{kn} - p\|
\]
\[
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\| + \alpha_{kn} \|y_{(k-1)n} - p\| + \beta_{kn} \|u_{kn} - p\|
\]
\[
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\| + \alpha_{kn} \|x_n - p\|
\]
\[
+ (\beta_n + \beta_{2n} + \ldots + \beta_{(k-1)n}) M + \beta_{kn} M
\]
\[
\leq (1 - \beta_{kn}) \|x_n - p\| + \beta_{kn} (\beta_1n + \beta_{2n} + \ldots + \beta_{(k-1)n}) M + \beta_{kn} M
\]
\[
\leq \|x_n - p\| + (\beta_1n + \beta_{2n} + \ldots + \beta_{(k-1)n} + \beta_{kn}) M
\]
\[
= \|x_n - p\| + \theta_n
\]

where \( \theta_n = M (\beta_1n + \beta_{2n} + \ldots + \beta_{(k-1)n} + \beta_{kn}) \). By assumption we have \( \sum_{n=1}^{\infty} \theta_n < \infty \). Thus by Lemma 4, \( \lim_{n \to \infty} \|x_n - p\| \) exist for any \( p \in F \).

Step 2. Show that \( \lim_{n \to \infty} \text{dist}(x_n, T_i(x_n)) = 0 \) for \( i = 1, 2, \ldots, k \).

Let \( p \in F \). From Step 1, we know that the sequences \( \{y_{1n}\}, \{y_{2n}\}, \ldots, \{y_{(k-1)n}\} \) and \( \{x_{n+1}\} \) are bounded. Therefore, we can find \( r > 0 \) depending on \( p \) such that \( y_{1n} - p, y_{2n} - p, \ldots, y_{(k-1)n} - p, x_{n+1} - p \in B_r(0) \) for all \( n \geq 1 \). As Step 1, there exists \( N > 0 \) such that
\[
\max \left\{ \sup_{n \geq 1} \|u_{1n} - p\|^2, \sup_{n \geq 1} \|u_{2n} - p\|^2, \ldots, \sup_{n \geq 1} \|u_{kn} - p\|^2 \right\} \leq N.
\]

From Lemma 5, we get
\[
\|y_{1n} - p\|^2 \leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\|^2 + \alpha_{1n} \|z_{1,1} - p\|^2 + \beta_{1n} \|u_{1n} - p\|^2
\]
\[
- \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi(\|x_n - z_{1,1}\|)
\]
\[
\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\|^2 + \alpha_{1n} \text{dist}(z_{1,1}, T_1(p))^2 + \beta_{1n} \|u_{1n} - p\|^2
\]
\[
- \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi(\|x_n - z_{1,1}\|)
\]
\[
\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\|^2 + \alpha_{1n} H(T_1(x_n), T_1(p))^2 + \beta_{1n} \|u_{1n} - p\|^2
\]
\[
- \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi(\|x_n - z_{1,1}\|)
\]
\[
\leq (1 - \beta_{1n}) \|x_n - p\|^2 + \beta_{1n} \|u_{1n} - p\|^2 - \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi(\|x_n - z_{1,1}\|)
\]
\[
\leq \|x_n - p\|^2 + \beta_{1n} N - \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi(\|x_n - z_{1,1}\|)
\]

and
\[
\|y_{2n} - p\|^2 \leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\|^2 + \alpha_{2n} \|z_{2,1} - p\|^2 + \beta_{2n} \|u_{2n} - p\|^2
\]
\[
- \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi(\|x_n - z_{2,1}\|)
\]
\[
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\|^2 + \alpha_{2n} \text{dist}(z_{2,1}, T_2(p))^2 + \beta_{2n} \|u_{2n} - p\|^2
\]
\[
- \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi(\|x_n - z_{2,1}\|)
\]
\[
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\|^2 + \alpha_{2n} H(T_2(y_{1n}), T_2(p))^2 + \beta_{2n} \|u_{2n} - p\|^2
\]
\[
- \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi(\|x_n - z_{2,1}\|)
\]
\[
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\|^2 + \alpha_{2n} \|x_n - p\|^2 + \beta_{1n} N - \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi(\|x_n - z_{1,1}\|)
\]
\[
+ \beta_{2n} \|u_{2n} - p\|^2 - \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi(\|x_n - z_{2,1}\|)
\]
\[
\leq \|x_n - p\|^2 + (\beta_{1n} + \beta_{2n}) N - \alpha_{1n} \alpha_{2n} (1 - \alpha_{1n} - \beta_{1n}) \varphi(\|x_n - z_{1,1}\|)
\]
\[
- \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi(\|x_n - z_{2,1}\|).
\]
It follows from Lemma 5 that
\[
\|y_{(k-1)n} - p\|^2 \leq \|x_n - p\|^2 + (\beta_{1n} + \beta_{2n} + \ldots + \beta_{(k-1)n}) N - \alpha_{1n} \alpha_{2n} \ldots \alpha_{(k-1)n} (1 - \alpha_{1n} - \beta_{1n}) \\
\varphi (\|x_n - z_{n,1}\|) - \alpha_{2n} \alpha_{3n} \ldots \alpha_{(k-1)n} (1 - \alpha_{2n} - \beta_{2n}) \varphi (\|x_n - z_{n,2}\|) \\
\ldots - \alpha_{(k-1)n} (1 - \alpha_{(k-1)n} - \beta_{(k-1)n}) \varphi (\|x_n - z_{n,k-1}\|)
\]
\[
\leq \|x_n - p\|^2 + (\beta_{1n} + \beta_{2n} + \ldots + \beta_{(k-1)n}) N \\
- \prod_{i=1}^{k-1} \alpha_{in} \left[ \sum_{i=1}^{k-1} (1 - \alpha_{in} - \beta_{in}) \varphi (\|x_n - z_{n,i}\|) \right].
\]
Again, we apply Lemma 5 to conclude that
\[
\|x_{n+1} - p\|^2 \leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\|^2 + \alpha_{kn} \|z_{n,k} - p\|^2 + \beta_{kn} \|u_{kn} - p\|^2 \\
- \alpha_{kn} (1 - \alpha_{kn} - \beta_{kn}) \varphi (\|x_n - z_{n,k}\|) \\
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\|^2 + \alpha_{kn} \text{dist}(z_{n,k}, T_k(p))^2 + \beta_{kn} \|u_{kn} - p\|^2 \\
- \alpha_{kn} (1 - \alpha_{kn} - \beta_{kn}) \varphi (\|x_n - z_{n,k}\|) \\
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\|^2 + \alpha_{kn} H(T_k(y_{(k-1)n}), T_k(p))^2 + \beta_{kn} \|u_{kn} - p\|^2 \\
- \alpha_{kn} (1 - \alpha_{kn} - \beta_{kn}) \varphi (\|x_n - z_{n,k}\|) \\
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\|^2 + \alpha_{kn} \|y_{(k-1)n} - p\|^2 + \beta_{kn} N \\
- \alpha_{kn} (1 - \alpha_{kn} - \beta_{kn}) \varphi (\|x_n - z_{n,k}\|) \\
\leq \|x_n - p\|^2 + (\beta_{1n} + \beta_{2n} + \ldots + \beta_{(k-1)n} + \beta_{kn}) N \\
- \prod_{i=1}^{k} \alpha_{in} \left[ \sum_{i=1}^{k} (1 - \alpha_{in} - \beta_{in}) \varphi (\|x_n - z_{n,i}\|) \right].
\]
Since \( \alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1) \) for \( i = 1, 2, \ldots, k \), we have
\[
d_k \sum_{i=1}^{k} (1 - b) \varphi (\|x_n - z_{n,i}\|) \leq \prod_{i=1}^{k} \alpha_{in} \left[ \sum_{i=1}^{k} (1 - \alpha_{in} - \beta_{in}) \varphi (\|x_n - z_{n,i}\|) \right] \\
\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (\beta_{1n} + \beta_{2n} + \ldots + \beta_{(k-1)n} + \beta_{kn}) N.
\]
This implies that
\[
\sum_{n=1}^{\infty} a_k \sum_{i=1}^{k} (1 - b) \varphi (\|x_n - z_{n,i}\|) \leq \|x_1 - p\|^2 + \sum_{n=1}^{\infty} (\beta_{1n} + \beta_{2n} + \ldots + \beta_{(k-1)n} + \beta_{kn}) N < \infty
\]
and hence \( \lim_{n \to \infty} \varphi (\|x_n - z_{n,i}\|) = 0 \). Since \( \varphi \) is continuous at 0 and is strictly increasing, we have
\[
\lim_{n \to \infty} \|x_n - z_{n,i}\| = 0.
\]
That is, for \( i = 1, 2, \ldots, k \)
\[
(6) \quad \lim_{n \to \infty} \|x_n - z_{n,1}\| = \lim_{n \to \infty} \|x_n - z_{n,2}\| = \ldots = \lim_{n \to \infty} \|x_n - z_{n,k}\| = 0.
\]
Also, using (A), (6) and \( \sum_{n=1}^{\infty} \beta_{in} < \infty \) for each \( i \), we have
\[
\lim_{n \to \infty} \|y_{1n} - x_n\| = \lim_{n \to \infty} (\alpha_{1n} \|z_{n,1} - x_n\| + \beta_{1n} \|u_{1n} - x_n\|) = 0, \\
\lim_{n \to \infty} \|y_{2n} - x_n\| = \lim_{n \to \infty} (\alpha_{2n} \|z_{n,2} - x_n\| + \beta_{2n} \|u_{2n} - x_n\|) = 0, \\
\vdots \\
\lim_{n \to \infty} \|x_{n+1} - x_n\| = \lim_{n \to \infty} (\alpha_{kn} \|z_{n,k} - x_n\| + \beta_{kn} \|u_{kn} - x_n\|) = 0.
Therefore by Lemma 3 we have

\[
\text{dist}(x_n, T_1(x_n)) \leq \text{dist}(x_n, T_1(x_n)) + H(T_1(x_n), T_1(x_n)) \\
\leq \text{dist}(x_n, T_1(x_n)) + \|x_n - x\| \\
\leq \|x_n - z_{n,1}\| \to 0 \text{ as } n \to \infty,
\]

and

\[
\text{dist}(x_n, T_2(x_n)) \leq \text{dist}(x_n, T_2(y_{1n})) + H(T_2(y_{1n}), T_2(x_n)) \\
\leq \text{dist}(x_n, T_2(y_{1n})) + 2\text{dist}(y_{1n}, T_2(y_{1n})) + \|y_{1n} - x\| \\
\leq 3\|y_{1n} - x\| + 3\text{dist}(x_n, T_2(y_{1n})) \\
\leq 3\|y_{1n} - x\| + 3\|x_n - z_{n,2}\| \to 0 \text{ as } n \to \infty.
\]

In a similar way, for \(i = 1, 2, \ldots, k\) we obtain that

\[
\lim_{n \to \infty} \text{dist}(x_n, T_i(x_n)) = 0.
\]

Step 3. Show that \(\{x_n\}\) converges strongly to \(q \in \mathcal{F}\).

From Step 2, we know that \(\lim_{n \to \infty} \text{dist}(x_n, T_i(x_n)) = 0\). By condition (II), we obtain that \(\lim_{n \to \infty} \text{dist}(x_n, \mathcal{F}) = 0\). Therefore, we can choose a subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) and a sequence \(\{p_j\}\) in \(\mathcal{F}\) such that for all \(j \in \mathbb{N}\)

\[
\|x_{n_j} - p_j\| < \frac{1}{2^j}.
\]

Therefore by inequality (5) we get

\[
\|x_{n_{j+1}} - p\| \leq \|x_{n_{j+1} - 1} - p\| + \theta_{n_{j+1} - 1} \\
\leq \|x_{n_{j+1} - 2} - p\| + \theta_{n_{j+1} - 2} + \theta_{n_{j+1} - 1} \\
\vdots \\
\leq \|x_{n_j} - p\| + \sum_{l=1}^{n_{j+1} - n_j - 1} \theta_{n_{j+l}}
\]

for all \(p \in \mathcal{F}\). This implies that

\[
\|x_{n_{j+1}} - p\| \leq \|x_{n_j} - p_j\| + \sum_{l=1}^{n_{j+1} - n_j - 1} \theta_{n_{j+l}} \\
< \frac{1}{2^j} + \sum_{l=1}^{n_{j+1} - n_j - 1} \theta_{n_{j+l}}.
\]

Now, we show that \(\{p_j\}\) is Cauchy sequence in \(E\). Note that

\[
\|p_{j+1} - p_j\| \leq \|p_{j+1} - x_{n_{j+1}}\| + \|x_{n_{j+1}} - p_j\| \\
< \frac{1}{2^{j+1}} + \frac{1}{2^j} + \sum_{l=1}^{n_{j+1} - n_j - 1} \theta_{n_{j+l}} \\
< \frac{1}{2^{j+1}} + \sum_{l=1}^{n_{j+1} - n_j - 1} \theta_{n_{j+l}}.
\]

Consequently, we conclude that \(\{p_j\}\) is Cauchy sequence in \(E\) and hence converges to \(q \in E\). Since for \(i = 1, 2, \ldots, k\)

\[
\text{dist}(p_j, T_i(q)) \leq H(T_i(p_j), T_i(q)) \leq \|p_j - q\|
\]
and \( p_j \to q \) as \( j \to \infty \), it follows that \( \text{dist}(q, T_i(q)) = 0 \) for \( i = 1, 2, \ldots, k \). Hence \( q \in \mathcal{F} \) and \( \{x_{n_j}\} \) converges strongly to \( q \). Since \( \lim_{n \to \infty} \|x_n - q\| \) exists, we conclude that \( \{x_n\} \) converges strongly to \( q \). 

Note that the condition (II) is weaker than the compactness of \( E \) and the hemi-compactness of the multi-valued mappings \( \{T_i : i = 1, 2, \ldots, k\} \), therefore we already have the following theorem.

**Theorem 7.** Let \( E \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \). Let \( T_i : E \to \text{CB}(E), \ (i = 1, 2, \ldots, k) \) be a finite family of multi-valued mappings satisfying the condition (C). Assume that \( \mathcal{F} = \cap_{i=1}^k F(T_i) \neq \emptyset \) and \( T_i(p) = \{p\}, (i = 1, 2, \ldots, k) \) for each \( p \in \mathcal{F} \). Let \( \{x_n\} \) be the iterative process defined by (A), and \( \alpha_n + \beta_n \in [a, b] \subset (0, 1) \) for \( i = 1, 2, \ldots, k \) and \( \sum_{n=1}^{\infty} \beta_n < \infty \) for each \( i \). Assume that either \( E \) is compact or one of the multi-valued mappings \( \{T_i : i = 1, 2, \ldots, k\} \) is hemi-compact. Then \( \{x_n\} \) converges strongly to a common fixed point of \( T_i \) for \( i = 1, 2, \ldots, k \).

**Proof.** From the proof of Theorem 6, we know that \( \lim_{n \to \infty} \text{dist}(x_n, T_i(x_n)) = 0 \) for each \( i \). We suppose that either \( E \) is compact or one of the multi-valued mappings \( \{T_i : i = 1, 2, \ldots, k\} \) is hemi-compact. Then there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) such that \( \lim_{j \to \infty} x_{n_j} = z \) for some \( z \in E \).

By Lemma 3, for \( i = 1, 2, \ldots, k \) we have

\[
\text{dist}(z, T_i(z)) \leq \|z - x_{n_j}\| + \text{dist}(x_{n_j}, T_i(z)) \\
\leq \|z - x_{n_j}\| + \text{dist}(x_{n_j}, T_i(x_{n_j})) + H(T_i(x_{n_j}), T_i(z)) \\
\leq 3\text{dist}(x_{n_j}, T_i(x_{n_j})) + 2\|z - x_{n_j}\| \to 0 \text{ as } j \to \infty.
\]

this implies that \( z \in \mathcal{F} \). Since \( \{x_{n_j}\} \) converges strongly to \( z \) and the limit \( \lim_{n \to \infty} \|x_n - z\| \) exists (by Theorem 6), it follows that \( \{x_n\} \) converges strongly to \( z \). 

**Theorem 8.** Let \( E \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \) with the Opial’s property. Let \( T_i : E \to K(E), \ (i = 1, 2, \ldots, k) \) be a finite family of multi-valued mappings satisfying the condition (C). Assume that \( \mathcal{F} = \cap_{i=1}^k F(T_i) \neq \emptyset \) and \( T_i(p) = \{p\}, (i = 1, 2, \ldots, k) \) for each \( p \in \mathcal{F} \). Let \( \{x_n\} \) be the iterative process defined by (A), and \( \alpha_n + \beta_n \in [a, b] \subset (0, 1) \) for \( i = 1, 2, \ldots, k \) and \( \sum_{n=1}^{\infty} \beta_n < \infty \) for each \( i \). Then \( \{x_n\} \) converges weakly to a common fixed point of \( T_i \) for \( i = 1, 2, \ldots, k \).

**Proof.** By Theorem 6, we have \( \{x_n\} \) is bounded and \( \lim_{n \to \infty} \text{dist}(x_n, T_i(x_n)) = 0 \) for each \( i \). Since \( E \) is uniformly convex, by passing to a subsequence we can assume that \( x_n \to q \) as \( n \to \infty \) for some \( q \in E \). We show that \( q \in \mathcal{F} \). Since \( T_1q \) is compact, we can choose \( y_n \in T_1q \) such that \( \|x_n - y_n\| = \text{dist}(x_n, T_1q) \). Since \( T_1q \) is compact the sequence \( \{y_n\} \) has a convergent subsequence \( \{y_{n_j}\} \) with \( \lim_{j \to \infty} y_{n_j} = z \in T_1q \). By Lemma 3, we have

\[
\text{dist}(x_{n_j}, T_1q) \leq \text{dist}(x_{n_j}, T_1(x_{n_j})) + H(T_1(x_{n_j}), T_1q) \\
\leq 3\text{dist}(x_{n_j}, T_1(x_{n_j})) + \|x_{n_j} - q\|.
\]

Note that

\[
\|x_{n_j} - z\| \leq \|x_{n_j} - y_{n_j}\| + \|y_{n_j} - z\| \\
\leq 3\text{dist}(x_{n_j}, T_1(x_{n_j})) + \|x_{n_j} - q\| + \|y_{n_j} - z\|.
\]

This implies that

\[
\limsup_{j \to \infty} \|x_{n_j} - z\| < \limsup_{j \to \infty} \|x_{n_j} - q\|.
\]

From the Opial’s property of Banach space \( X \), we have \( q = z \in T_1q \). Similarly, it can be shown that \( q \in T_iq \) for \( i = 2, 3, \ldots, k \). Now we prove that \( \{x_n\} \) has a unique weak subsequential limit in \( \mathcal{F} \). To prove this, let \( w \) and \( z \) be weak limits of the subsequences \( \{x_{n_j}\} \) and \( \{x_{n_m}\} \) of \( \{x_n\} \),
respectively and \( w \neq z \). As above, \( w, z \in \mathcal{F} \), and hence by Theorem 6, the limits \( \lim_{n \to \infty} \| x_n - w \| \) and \( \lim_{n \to \infty} \| x_n - z \| \) exist. Then by Opial’s property,

\[
\lim_{n \to \infty} \| x_n - w \| = \lim_{j \to \infty} \| x_{n_j} - w \| < \lim_{j \to \infty} \| x_{n_j} - z \| = \lim_{n \to \infty} \| x_n - z \| = \lim_{m \to \infty} \| x_{n_m} - z \| < \lim_{m \to \infty} \| x_{n_m} - w \| = \lim_{n \to \infty} \| x_n - w \|
\]

which is a contradiction. Therefore \( \{ x_n \} \) converges weakly to a point in \( \mathcal{F} \). \( \square \)

From Lemma 1, we know that if \( T \) is a multi-valued nonexpansive mapping, then \( T \) satisfies the condition (C). So we have the following results:

**Corollary 9.** Let \( E \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \). Let \( T_i : E \to CB(E), \ (i = 1, 2, \ldots, k) \) be a finite family of multi-valued nonexpansive mappings. Assume that \( \mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset \) and \( T_i(p) = \{ p \}, \ (i = 1, 2, \ldots, k) \) for each \( p \in \mathcal{F} \). Let \( \{ x_n \} \) be the iterative process defined by (A), and \( \alpha_n + \beta_n \in [a, b] \subset (0, 1) \) for \( i = 1, 2, \ldots, k \) and \( \sum_{n=1}^{\infty} \beta_n < \infty \) for each \( i \). Assume that \( T_i (i = 1, 2, \ldots, k) \) satisfying the condition \( (II) \). Then \( \{ x_n \} \) converges strongly to a common fixed point of \( T_i \) for \( i = 1, 2, \ldots, k \).

**Corollary 10.** Let \( E \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \). Let \( T_i : E \to CB(E), \ (i = 1, 2, \ldots, k) \) be a finite family of multi-valued nonexpansive mappings. Assume that \( \mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset \) and \( T_i(p) = \{ p \}, \ (i = 1, 2, \ldots, k) \) for each \( p \in \mathcal{F} \). Let \( \{ x_n \} \) be the iterative process defined by (A), and \( \alpha_n + \beta_n \in [a, b] \subset (0, 1) \) for \( i = 1, 2, \ldots, k \) and \( \sum_{n=1}^{\infty} \beta_n < \infty \) for each \( i \). Assume that either \( E \) is compact or one of the multi-valued mappings \( \{ T_i (i = 1, 2, \ldots, k) \} \) is hemi-compact. Then \( \{ x_n \} \) converges strongly to a common fixed point of \( T_i \) for \( i = 1, 2, \ldots, k \).

**Corollary 11.** Let \( E \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \) with the Opial’s property. Let \( T_i : E \to K(E), \ (i = 1, 2, \ldots, k) \) be a finite family of multi-valued nonexpansive mappings. Assume that \( \mathcal{F} = \cap_{i=1}^{k} F(T_i) \neq \emptyset \) and \( T_i(p) = \{ p \}, \ (i = 1, 2, \ldots, k) \) for each \( p \in \mathcal{F} \). Let \( \{ x_n \} \) be the iterative process defined by (A), and \( \alpha_n + \beta_n \in [a, b] \subset (0, 1) \) for \( i = 1, 2, \ldots, k \) and \( \sum_{n=1}^{\infty} \beta_n < \infty \) for each \( i \). Then \( \{ x_n \} \) converges weakly to a common fixed point of \( T_i \) for \( i = 1, 2, \ldots, k \).

We know intend to remove the restriction that \( T_i(p) = \{ p \} \) for each \( p \in \mathcal{F} \). We define the following iteration process.

**(B):** Let \( E \) be a nonempty convex subset of a Banach space \( X \) and \( T_i : E \to CB(E) (i = 1, 2, \ldots, k) \) be a finite family of multi-valued mappings and

\[
P_{T_i}(x) = \{ y \in T_i(x) : \| x - y \| = \text{dist} (x, T_i(x)) \}.
\]

Then for \( x_1 \in E \), we consider the following iterative process:

\[
\begin{align*}
y_{1n} &= (1 - \alpha_{1n} - \beta_{1n}) x_n + \alpha_{1n} z_{n,1} + \beta_{1n} u_{1n}, \quad n \geq 1, \\
y_{2n} &= (1 - \alpha_{2n} - \beta_{2n}) x_n + \alpha_{2n} z_{n,2} + \beta_{2n} u_{2n}, \quad n \geq 1, \\
& \vdots \\
y_{(k-1)n} &= (1 - \alpha_{(k-1)n} - \beta_{(k-1)n}) x_n + \alpha_{(k-1)n} z_{n,(k-1)} + \beta_{(k-1)n} u_{(k-1)n}, \quad n \geq 1, \\
x_{n+1} &= (1 - \alpha_{kn} - \beta_{kn}) x_n + \alpha_{kn} z_{n,k} + \beta_{kn} u_{kn}, \quad n \geq 1,
\end{align*}
\]

where \( z_{n,1} \in P_{T_1}(x_n) \) and \( z_{n,j} \in P_{T_i}(y_{i(j-1)n}) \) for \( i = 2, 3, \ldots, k \) and \( \{ \alpha_{in} \}, \{ \beta_{in} \} \in [0, 1] \) and \( \{ u_{in} \} \) are bounded sequences in \( E \).

**Theorem 12.** Let \( E \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \). Let \( T_i : E \to P(E), \ (i = 1, 2, \ldots, k) \) be a finite family of multi-valued mappings such that \( P_{T_i} \) satisfying the condition \( (C) \) for \( i = 1, 2, \ldots, k \). Let \( \{ x_n \} \) be the iterative process defined by (B),
and \( \alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1) \) for \( i = 1, 2, ..., k \) and \( \sum_{i=1}^{\infty} \beta_{in} < \infty \) for each \( i \). Assume that \( T_i \) \( (i = 1, 2, ..., k) \) satisfying the condition (II) and \( F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset \). Then \( \{x_n\} \) converges strongly to a common fixed point of \( T_i \) for \( i = 1, 2, ..., k \).

**Proof.** Firstly, we show that \( \lim_{n \to \infty} \|x_n - p\| \) exists for any \( p \in F \). Let \( p \in F \). Then, for \( i = 1, 2, ..., k \) we have \( p \in P_{T_i}(p) = \{p\} \). Since \( \{u_{in}\} \) are bounded for each \( i \), there exists \( M > 0 \) such that

\[
\max \left\{ \sup_{n \geq 1} \|u_{1n} - p\|, \sup_{n \geq 1} \|u_{2n} - p\|, ..., \sup_{n \geq 1} \|u_{kn} - p\| \right\} \leq M.
\]

From the iteration process (A) and quasi-nonexpansiveness of \( T_i \) \( (i = 1, 2, ..., k) \) we obtain

\[
\|y_{1n} - p\| \leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} \|z_{n,1} - p\| + \beta_{1n} \|u_{1n} - p\|
\]

\[
\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} \text{dist}(z_{n,1}, P_{T_1}(p)) + \beta_{1n} \|u_{1n} - p\|
\]

\[
\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\| + \alpha_{1n} H(P_{T_1}(x_n), P_{T_1}(p)) + \beta_{1n} \|u_{1n} - p\|
\]

\[
\leq (1 - \beta_{1n}) \|x_n - p\| + \alpha_{1n} \|x_n - p\| + \beta_{1n} \|u_{1n} - p\|
\]

\[
= (1 - \beta_{1n}) \|x_n - p\| + \beta_{1n} M
\]

and

\[
\|y_{2n} - p\| \leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \|z_{n,2} - p\| + \beta_{2n} \|u_{2n} - p\|
\]

\[
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \text{dist}(z_{n,2}, P_{T_2}(p)) + \beta_{2n} \|u_{2n} - p\|
\]

\[
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} H(P_{T_2}(y_{1n}), P_{T_2}(p)) + \beta_{2n} \|u_{2n} - p\|
\]

\[
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \|y_{1n} - p\| + \beta_{2n} \|u_{2n} - p\|
\]

\[
\leq (1 - \beta_{2n}) \|x_n - p\| + \alpha_{2n} \|x_n - p\| + \beta_{2n} M + \beta_{2n} M
\]

\[
\leq \|x_n - p\| + (\beta_{1n} + \beta_{2n}) M.
\]

Similarly, we get

\[
\|y_{(k-1)n} - p\| \leq \|x_n - p\| + (\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n}) M,
\]

and

\[
\|x_{n+1} - p\| \leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\| + \alpha_{kn} \|z_{n,i} - p\| + \beta_{kn} \|u_{kn} - p\|
\]

\[
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\| + \alpha_{kn} \text{dist}(z_{n,k}, P_{T_k}(p)) + \beta_{kn} \|u_{kn} - p\|
\]

\[
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\| + \alpha_{kn} H(P_{T_k}(y_{(k-1)n}), P_{T_k}(p)) + \beta_{kn} \|u_{kn} - p\|
\]

\[
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\| + \alpha_{kn} \|y_{(k-1)n} - p\| + \beta_{kn} \|u_{kn} - p\|
\]

\[
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\| + \alpha_{kn} \|x_n - p\| + \beta_{kn} M + \beta_{kn} M
\]

\[
\leq \|x_n - p\| + (\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n}) M + \beta_{kn} M
\]

\[
= \|x_n - p\| + \alpha_{kn} (\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n}) M + \beta_{kn} M
\]

\[
\leq \|x_n - p\| + (\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n}) M + \beta_{kn} M
\]

\[
\leq \|x_n - p\| + \theta_n
\]

where \( \theta_n = M (\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n} + \beta_{kn}) \). Since \( \sum_{i=1}^{\infty} \beta_{in} < \infty \) for each \( i \), we have \( \sum_{i=1}^{\infty} \theta_n < \infty \). Therefore from Lemma 4, \( \lim_{n \to \infty} \|x_n - p\| \) exists for any \( p \in F \). Since the sequences \( \{y_{1n}\}, \{y_{2n}\}, ..., \{y_{(k-1)n}\} \) and \( \{x_{n+1}\} \) are bounded, we can find \( r > 0 \) depending on \( p \) such that \( y_{1n} - p, y_{2n} - p, ..., y_{(k-1)n} - p, x_{n+1} - p \in B_r(0) \) for all \( n \geq 1 \). Also, since \( \{u_{in}\} \) are bounded for each \( i \), there exists \( N > 0 \) such that

\[
\max \left\{ \sup_{n \geq 1} \|u_{1n} - p\|^2, \sup_{n \geq 1} \|u_{2n} - p\|^2, ..., \sup_{n \geq 1} \|u_{kn} - p\|^2 \right\} \leq N.
\]
Using Lemma 5, we have

\[
\|y_{1n} - p\|^2 \leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\|^2 + \alpha_{1n} \|z_{n,1} - p\|^2 + \beta_{1n} \|u_{1n} - p\|^2 \\
- \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi (\|x_n - z_{n,1}\|)
\]

\[
\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\|^2 + \alpha_{1n} \text{dist} (z_{n,1}, P_{T_1} (p))^2 + \beta_{1n} \|u_{1n} - p\|^2 \\
- \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi (\|x_n - z_{n,1}\|)
\]

\[
\leq (1 - \alpha_{1n} - \beta_{1n}) \|x_n - p\|^2 + \alpha_{1n} \mathcal{H} (P_{T_1} (x_n), P_{T_1} (p))^2 + \beta_{1n} \|u_{1n} - p\|^2 \\
- \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi (\|x_n - z_{n,1}\|)
\]

\[
\leq (1 - \beta_{1n}) \|x_n - p\|^2 + \beta_{1n} \|u_{1n} - p\|^2 - \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi (\|x_n - z_{n,1}\|)
\]

\[
\leq \|x_n - p\|^2 + \beta_{1n} N - \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \varphi (\|x_n - z_{n,1}\|)
\]

and

\[
\|y_{2n} - p\|^2 \leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\|^2 + \alpha_{2n} \|z_{n,2} - p\|^2 + \beta_{2n} \|u_{2n} - p\|^2 \\
- \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi (\|x_n - z_{n,2}\|)
\]

\[
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\|^2 + \alpha_{2n} \text{dist} (z_{n,2}, P_{T_2} (p))^2 + \beta_{2n} \|u_{2n} - p\|^2 \\
- \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi (\|x_n - z_{n,2}\|)
\]

\[
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\|^2 + \alpha_{2n} \mathcal{H} (P_{T_2} (y_{1n}), P_{T_2} (p))^2 + \beta_{2n} \|u_{2n} - p\|^2 \\
- \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi (\|x_n - z_{n,2}\|)
\]

\[
\leq (1 - \alpha_{2n} - \beta_{2n}) \|x_n - p\|^2 + \alpha_{2n} \|x_n - p\|^2 + \beta_{1n} N - \alpha_{1n} (1 - \alpha_{1n} - \beta_{1n}) \\
\varphi (\|x_n - z_{n,1}\|) + \beta_{2n} \|u_{2n} - p\|^2 - \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi (\|x_n - z_{n,2}\|)
\]

\[
\leq (1 - \beta_{2n}) \|x_n - p\|^2 + \alpha_{2n} \beta_{1n} N - \alpha_{1n} \alpha_{2n} (1 - \alpha_{1n} - \beta_{1n}) \varphi (\|x_n - z_{n,1}\|) \\
+ \beta_{2n} N - \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi (\|x_n - z_{n,2}\|)
\]

\[
\leq \|x_n - p\|^2 + (\beta_{1n} + \beta_{2n}) N - \alpha_{1n} \alpha_{2n} (1 - \alpha_{1n} - \beta_{1n}) \varphi (\|x_n - z_{n,1}\|) \\
- \alpha_{2n} (1 - \alpha_{2n} - \beta_{2n}) \varphi (\|x_n - z_{n,2}\|).
\]

From Lemma 5, we obtain

\[
\|y_{(k-1)n} - p\|^2 \leq \|x_n - p\|^2 + (\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n}) N - \alpha_{1n} \alpha_{2n} ... \alpha_{(k-1)n} (1 - \alpha_{1n} - \beta_{1n}) \\
\varphi (\|x_n - z_{n,1}\|) - \alpha_{2n} \alpha_{3n} ... \alpha_{(k-1)n} (1 - \alpha_{2n} - \beta_{2n}) \varphi (\|x_n - z_{n,2}\|) \\
- ... - \alpha_{(k-1)n} (1 - \alpha_{(k-1)n} - \beta_{(k-1)n}) \varphi (\|x_n - z_{n,k-1}\|)
\]

\[
\leq \|x_n - p\|^2 + (\beta_{1n} + \beta_{2n} + ... + \beta_{(k-1)n}) N \\
- \prod_{i=1}^{k-1} \alpha_{in} \left[ \sum_{i=1}^{k-1} (1 - \alpha_{in} - \beta_{in}) \varphi (\|x_n - z_{n,i}\|) \right].
\]
Again, we apply Lemma 5 to conclude that

\[
\|x_{n+1} - p\|^2 \leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\|^2 + \alpha_{kn} \|z_{n,k} - p\|^2 + \beta_{kn} \|u_{kn} - p\|^2
\]

\[
- \alpha_{kn} (1 - \alpha_{kn} - \beta_{kn}) \varphi (\|x_n - z_{n,k}\|)
\]

\[
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\|^2 + \alpha_{kn} \text{dist} (z_{n,k}, P_{T_k} (p))^2 + \beta_{kn} \|u_{kn} - p\|^2
\]

\[
- \alpha_{kn} (1 - \alpha_{kn} - \beta_{kn}) \varphi (\|x_n - z_{n,k}\|)
\]

\[
\leq (1 - \alpha_{kn} - \beta_{kn}) \|x_n - p\|^2 + \alpha_{kn} H (P_{T_k} (y_{(k-1)n}), P_{T_k} (p))^2 + \beta_{kn} \|u_{kn} - p\|^2
\]

\[
- \alpha_{kn} (1 - \alpha_{kn} - \beta_{kn}) \varphi (\|x_n - z_{n,k}\|)
\]

\[
\leq \|x_n - p\|^2 + (\beta_1 + \beta_2 + \ldots + \beta_{(k-1)n} + \beta_{kn}) N
\]

\[
- \prod_{i=1}^k \alpha_{in} \left[ \sum_{i=1}^k (1 - \alpha_{in} - \beta_{in}) \varphi (\|x_n - z_{n,i}\|) \right]
\]

Since \( \alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1) \) for \( i = 1, 2, \ldots, k \), we have

\[
a^k \sum_{i=1}^k (1 - b) \varphi (\|x_n - z_{n,i}\|) \leq \prod_{i=1}^k \alpha_{in} \left[ \sum_{i=1}^k (1 - \alpha_{in} - \beta_{in}) \varphi (\|x_n - z_{n,i}\|) \right]
\]

\[
\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (\beta_1 + \beta_2 + \ldots + \beta_{(k-1)n} + \beta_{kn}) N
\]

This implies that

\[
\sum_{n=1}^\infty \left[ a^k \sum_{i=1}^k (1 - b) \varphi (\|x_n - z_{n,i}\|) \right] \leq \|x - p\|^2 + \sum_{n=1}^\infty (\beta_1 + \beta_2 + \ldots + \beta_{(k-1)n} + \beta_{kn}) N < \infty
\]

and hence \( \lim_{n \to \infty} \varphi (\|x_n - z_{n,i}\|) = 0 \). Since \( \varphi \) is continuous at 0 and is strictly increasing, we have

\[
\lim_{n \to \infty} \|x_n - z_{n,i}\| = 0.
\]

As in the proof of Theorem 6, we obtain that \( \lim_{n \to \infty} \text{dist} (x_n, \mathcal{F}) = 0 \). Therefore, we can choose a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) and a sequence \( \{p_j\} \) in \( \mathcal{F} \) such that for all \( j \in \mathbb{N} \)

\[
\|x_{n_j} - p_j\| < \frac{1}{2^j}.
\]

As in the proof of Theorem 6, \( \{p_j\} \) is Cauchy sequence in \( E \) and hence converges to \( q \in E \). Since for \( i = 1, 2, \ldots, k \)

\[
dist (p_j, T_i (q)) \leq dist (p_j, P_{T_i} (p_j)) \leq H (P_{T_i} (p_j), P_{T_i} (q)) \leq \|p_j - q\|
\]

and \( p_j \to q \) as \( j \to \infty \), it follows that \( \text{dist} (q, T_i (q)) = 0 \) for \( i = 1, 2, \ldots, k \). Hence \( q \in \mathcal{F} \) and \( \{x_{n_j}\} \)

converges strongly to \( q \). Since \( \lim_{n \to \infty} \|x_n - q\| \) exists, we conclude that \( \{x_n\} \) converges strongly to \( q \).

Now, using Lemma 1, we obtain the following results:

**Corollary 13.** Let \( E \) be a nonempty closed convex subset of a uniformly convex Banach space \( X \). Let \( T_i : E \to P (E) \), \( i = 1, 2, \ldots, k \) be a finite family of multi-valued mappings such that \( P_{T_i} \) is nonexpansive for \( i = 1, 2, \ldots, k \). Let \( \{x_n\} \) be the iterative process defined by (B), and \( \alpha_{in} + \beta_{in} \in [a, b] \subset (0, 1) \) for \( i = 1, 2, \ldots, k \) and \( \sum_{i=1}^\infty \beta_{in} < \infty \) for each \( i \). Assume that \( T_i \), \( i = 1, 2, \ldots, k \)

satisfying the condition (II) and \( \mathcal{F} = \bigcap_{i=1}^k F (T_i) \neq \emptyset \). Then \( \{x_n\} \) converges strongly to a common fixed point of \( T_i \) for \( i = 1, 2, \ldots, k \).
Corollary 14. Let $E$ be a nonempty closed convex subset of a uniformly convex Banach space $X$. Let $T_i : E \rightarrow P(E)$, $(i = 1, 2, ..., k)$ be a finite family of multi-valued mappings such that $P_{T_i}$ is nonexpansive for $i = 1, 2, ..., k$. Let $\{x_n\}$ be the iterative process defined by (B), and $\alpha_n + \beta_n \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., k$ and $\sum_{n=1}^{\infty} \beta_n < \infty$ for each $i$. Assume that one of the multi-valued mappings $\{T_i : i = 1, 2, ..., k\}$ is hemi-compact and $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a common fixed point of $T_i$ for $i = 1, 2, ..., k$.

Corollary 15. Let $E$ be a nonempty closed convex subset of a uniformly convex Banach space $X$ with the Opial’s property. Let $T_i : E \rightarrow P(E)$, $(i = 1, 2, ..., k)$ be a finite family of multi-valued mappings such that $P_{T_i}$ is nonexpansive for $i = 1, 2, ..., k$. Let $\{x_n\}$ be the iterative process defined by (B), and $\alpha_n + \beta_n \in [a, b] \subset (0, 1)$ for $i = 1, 2, ..., k$ and $\sum_{n=1}^{\infty} \beta_n < \infty$ for each $i$. Assume that $I - T_i$ is demi-closed at $0$ for each $i = 1, 2, ..., k$ and $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. Then $\{x_n\}$ converges weakly to a common fixed point of $T_i$ for $i = 1, 2, ..., k$.

REFERENCES


Department of Mathematics, Ataturk University, Erzurum 25240, Turkey
E-mail address: nazli.kadioglu@atauni.edu.tr, isayildirim@atauni.edu.tr